Shapely monads for graphical calculi

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Outline

1. Introduction
2. Preliminaries
3. Operads
4. Graphical operads
5. Shapely monads
Mathematical motivation

Certain algebraic structures with
  - obvious graphical intuition;
  - tedious formal definition.

E.g., operads, properads, polycategories, PROPs, and variants.
Computer science motivation

Graphical calculi with
- obvious graphical intuition;
- tedious formal definition;
- involved or non-existent notion of model.

E.g., took quite long to work out for proof nets\(^1\)!

**Example**

Interaction nets, (multiplicative) proof nets, bigraphs, ZX-calculus.

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Contributions

- Make graphical intuition rigorous thanks to presheaf theory.
- $\mapsto$ Alternative definition of
  
  **maths:** the algebraic structure in question
  **comp. sci.:** a notion of model for the graphical calculus in question.

- View old definition as economical characterisation:

<table>
<thead>
<tr>
<th></th>
<th>old definition</th>
<th>new definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>statement</td>
<td>hard</td>
<td>easy</td>
</tr>
<tr>
<td>construction</td>
<td>easy</td>
<td>hard</td>
</tr>
</tbody>
</table>
Posing the problem categorically

\[ \text{presheaves} \leadsto \text{endofunctor } B \leadsto \text{monad } T \leadsto \text{T-algebras} \]

pictures \leadsto \text{algebraic structures}

Need to explain these terms, at least intuitively.

- Rightmost part: standard categorical approach to algebra.
- Difficult part in red!
Definition

**Objects**, and **morphisms** between them.

Example

<table>
<thead>
<tr>
<th></th>
<th>Objects</th>
<th>Morphisms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set</td>
<td>Sets</td>
<td>Functions</td>
</tr>
<tr>
<td>Mon</td>
<td>Monoids</td>
<td>Monoid homomorphisms</td>
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<td>Grp</td>
<td>Groups</td>
<td>Group homomorphisms</td>
</tr>
<tr>
<td>. . .</td>
<td>. . .</td>
<td>. . .</td>
</tr>
</tbody>
</table>
Functors

Definition

Functor = morphism of categories.

Example

- **Action on objects:**
  \[ L(X) = \sum_n X^n \]
  = sequences of elements of \( X \),
  = free monoid on \( X \).

- **Multiplication:**
  \[(x_1, \ldots, x_n), (x_{n+1}, \ldots, x_p) \mapsto (x_1, \ldots, x_p).\]

- **Action on morphisms:**
  \[ L(X \xrightarrow{f} Y): L(X) \rightarrow L(Y) \]
  \[(x_1, \ldots, x_n) \mapsto (f(x_1), \ldots, f(x_n)).\]

- **Other example:**
  \[ U(M) = |M|, \text{ carrier of } M. \]
Monads

Definition

Monad = endofunctor + structure.

Example

- Composite $T = U \circ L$.
- $T(X) =$ free monoid viewed as a set.
- $T$ is a monad.
Crucial point I: algebraic structures = algebras for a monad

**$T$-algebra**

$T$-algebra = morphism

\[

t(X)
\]

with easy conditions.

**Example: previous $T$**

- $T(X)$ = free monoid viewed as a set.
- So $m$ maps sequences $(x_1, \ldots, x_n)$ to elements.
- Thought of as multiplication.

Example $T$-algebra:

\[
m: T(\mathbb{N}) \rightarrow \mathbb{N}
\]

\[
(n_1, \ldots, n_p) \mapsto \sum_i n_i.
\]
Morphisms of algebras

Morphisms of $T$-algebras

$f(m(x_1, \ldots, x_n)) = m'(f(x_1), \ldots, f(x_n))$.

Morphism = structure-preserving map.

Proposition (in the monoids example)

$T$-algebras form a category $T\text{-Alg}$, equivalent to Mon.

Moral (standard, but very important!)

Algebraic structure (monoids) $\leftrightarrow$ monad $T$.

$T$ describes ‘free’ algebraic structures.
**Other examples on sets**

<table>
<thead>
<tr>
<th>Algebraic structure</th>
<th>$T(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monoids</td>
<td>$\sum_n X^n$</td>
</tr>
<tr>
<td>Commutative monoids</td>
<td>$\sum_n X^n / \mathfrak{S}_n$</td>
</tr>
<tr>
<td>Rings, modules, algebras, . . .</td>
<td>...</td>
</tr>
<tr>
<td>Complete semi-lattices</td>
<td>$\mathcal{P}(X)$</td>
</tr>
</tbody>
</table>

Non-example: fields, as there are no free fields over a set.
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Running example: (nonsymmetric, coloured) operads.

Well-known case: $T$ already known!

Result specialises to: characterisation of $T$ as a free shapely monad.

\[
\text{family of presheaves} \xrightarrow{\sim} \text{endofunctor } B \xrightarrow{\sim} \text{monad } T \xrightarrow{\sim} T\text{-algebras}
\]

\[
\text{pictures} \quad \text{algebraic structures}
\]
From pictures to presheaves

- Running example: (nonsymmetric, coloured) operads.
- Well-known case: $T$ already known!
- Result specialises to: characterisation of $T$ as a free shapely monad.

family of multigraphs $\rightsquigarrow$ endofunctor $B \rightsquigarrow$ monad $T \rightsquigarrow \rightsquigarrow T$-algebras

$\downarrow$

pictures $\downarrow$ algebraic structures
Multigraphs

Multigraph $X \approx$ graph whose edges may have several sources.

Diagram

- $X_0$: vertices;
- $X_n$: edges with $n$ sources;
- $s_{n,i}(e)$: $i$th source of $n$-ary $e$;
- $t_n(e)$: target of $e$. 
Example multigraph

- $X_\ast = \{a, b, c, d, e\}$,
- $X_2 = \{x, y\}$,
- $X_n = \emptyset$ otherwise,
- $t_2(x) = x \cdot t = a$ (notation!),
- $x \cdot s_1 = b$, $x \cdot s_2 = c$, $y \cdot t = c$,
- $y \cdot s_1 = d$, $y \cdot s_2 = e$. 
Category of multigraphs

Morphism = map preserving target and sources.

Proposition

Multigraphs form a category $\text{MGph}$. 
Intuitive definition

A *(nonsymmetric, coloured) operad* (in sets) $\mathcal{O}$ is a multigraph $\mathcal{O}$ with ‘plugging’, e.g., for all $x \in \mathcal{O}_2$ and $y \in \mathcal{O}_3$ with $y \cdot t = x \cdot s_1$, one may form

![Diagram of operad](image)

in $\mathcal{O}_4$.

**Notation**

Denoted by $x \circ^{2,3}_1 y$. 
Plugging should satisfy obvious graphical axioms, e.g.,
Dreadful glimpses of standard definition

**Definition**

A *(nonsymmetric, coloured) operad* (in sets) is

- a multigraph $\mathcal{O}$, together with
- for all $m, n, i, x \in \mathcal{O}_m$ and $y \in \mathcal{O}_n$ such that $x \cdot s_i = y \cdot t$, an element
  $$x \circ_{i}^{m,n} y \in \mathcal{O}_{m+n-1};$$
- for all $a \in \mathcal{O}_\star$, an element $id_a \in \mathcal{O}_1$;
- satisfying axioms like
  $$(x \circ_{i}^{m,n} y) \circ_{j}^{m+n-1,p} z = \begin{cases} (x \circ_{j}^{m,p} z) \circ_{i+p-1}^{m+p-1,n} y & \text{(if } j < i) \\ x \circ_{i}^{m,n+p-1} (y \circ_{j-i+1}^{n,p} z) & \text{(if } i \leq j < i + n) \end{cases}$$
  for all $x \in \mathcal{O}_m$, $y \in \mathcal{O}_n$, $z \in \mathcal{O}_p$. 
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Endofunctors from multigraphs

family of multigraphs \(\leadsto\) endofunctor \(B\) \(\leadsto\) monad \(T\) \(\leadsto\) \(T\)-algebras

pictures \(\leadsto\) algebraic structures
Crucial point II:
arguments for composition = multigraph morphisms

• Recall the picture for composition in $\mathcal{O}$, on the right.
• View it as a multigraph, say $X$.

(Morphisms $X \to \mathcal{O}$) $\iff$ (choices of $(x, y)$):
  • $x \in \mathcal{O}_2$ and $y \in \mathcal{O}_3$,
  • such that $x \cdot s_1 = y \cdot t$.

= potential arguments for $\circ_{1,3}^{2,3}$ if it existed.
Arities

Definition (Basic arities)

- $X$ is the arity of $o_{1}^{2,3}$.
- Obvious generalisation: $X_{i}^{m,n}$ is the arity of $o_{i}^{m,n}$.
- Similarly, arity of $id$: multigraph with just one vertex (wire).
Making sense of $h_X$-algebras

- Recall our example multigraph $X$ on the right.
- Consider the functor $h_X : \text{MGph} \to \text{MGph}$ defined by:
  - $h_X(Y)_* = Y_*,$
  - $h_X(Y)_4 = \text{MGph}(X, Y),$ the set of multigraph morphisms from $X$ to $Y$,
  - $h_X(Y)_n = \emptyset$ for $n \neq 4$.
- So $h_X(Y)_4 = \{(x', y') \in Y_2 \times Y_3 \mid x' \cdot s_1 = y' \cdot t\}.$
- An algebra $h_X(Y) \to Y$ is determined by:
  - a multigraph $Y$,
  - plus a map $h_X(Y)_4 \to Y_4$, i.e.,
  - an interpretation of $\circ_1^{2,3}!$

Summary

Multigraph $X \rightsquigarrow$ functor which specifies an operation of arity $X$.

I.e., algebras have such an operation.
The monad from derived arities

family of multigraphs $\leadsto$ endofunctor $B \leadsto$ monad $T \leadsto T$-algebras

pictures $\leadsto$ algebraic structures
Graphical definition of operads

Need to define arities for all derived operations:

**Definition**
Let $T_n$ denote the class of planar trees with $n$ leaves.

Define $T : \text{MGph} \rightarrow \text{MGph}$ by:

- $T(Y)_\star = Y_\star$,
- $T(Y)_n = \sum_{X \in T_n} \text{MGph}(X, Y)$, the set of multigraph morphisms from some $n$-ary tree $X$ to $Y$.

**Lemma**
*The functor $T$ is a monad on MGph.*

**Theorem**
*Operads are equivalent to $T$-algebras.*
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Generating monads

- Family of multigraphs \( \rightsquigarrow \) endofunctor \( B \rightsquigarrow \) monad \( T \rightsquigarrow \) \( T \)-algebras

- Goal: generate \( T \) automatically from basic arities.

- Compositions \( X_i^{n,m} \).
- Identities \( I_a \).
Signature for operads

Definition
Let $\mathcal{B}_n$ denote the set of basic arities with $n$ leaves.

Intuition: filiform trees of depth 2.

Define $B : \text{MGph} \to \text{MGph}$ by:

- $B(Y)_* = Y_*,$
- $B(Y)_n = \sum_{X \in \mathcal{B}_n} \text{MGph}(X, Y),$ the set of multigraph morphisms from some $n$-ary basic arity $X$ to $Y$.

Question: how to generate $T$ from $B$?
Naive attempt

Well-known correspondence

Endofunctors on $\text{MGph}$ \xrightarrow{\mathcal{M}} \text{Monads on } \text{MGph}.

Miss!

$\mathcal{M}(B) \not\cong T$. 

End of the page
Reason

$\mathcal{M}(B)$-algebras do not satisfy any of the axioms!

Which monads do enforce them? Shapely ones!
Shapely monads

Subcategory

\[
\text{Framed}(MGph) \subseteq \text{Cell}(MGph) \subseteq \text{Analytic}(MGph) \subseteq \text{Endo}(MGph).
\]

- Stable under composition.
- Has a terminal object \( \top \), automatically a monad.

**Definition**

Shapely = subfunctor of \( \top \) in Framed(MGph).

Graphical calculus = shapely monad.

Intuition: at most one operation of each arity.
**Generation result**

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**Theorem**

\[ T = \bigcup_n (\text{id} \cup B)^n \text{ is the free shapely monad over } B. \]

\( B \cdot B \) denotes the **image** of \( B \circ B: B \circ B \rightarrow B \cdot B \leftarrow T \).
Illustration of $B \cdot B$
General result

- Consider any presheaf category with a subterminal object $\top$.
  
  At most one morphism from any object to $\top$.

- Consider $\top$-analytic functors, i.e., analytic functors with a map to $\top$.
- Suppose they are stable under composition.
- Example: framed endofunctors.

**Definition**

**Shapely functor** = subfunctor of $\top$.

**Theorem**

The free shapely monad on a shapely endofunctor $B$ is $\bigcup_n (id \cup B)^n$. 
Applications

- Characterisation of the monads for polycategories, properads, PROPs, etc, as free shapely monads.
- Definition of free shapely monads for interaction nets and fragments of proof nets.
If you wonder what a model for your graphical calculus is, you could try to:

- Formalise pictures as presheaves.
- Derive monad automatically from them.
- Then work on an intelligible characterisation of algebras.
Perspectives

- Further applications, e.g., to PROP rewriting (started long ago with Adrien D., idle).
- Framed functors: 2-levels, level 1 fixed. Generalisation?
- Notion of representable algebra, as in representable operad.
- Existence of a tensor product.
- Notion of weak algebra (suggested by Kris W.), as in $\infty$-operad.
- Generalise to not strictly shapely, e.g., proof net boxes.
Thanks!
Shapely functors: intuition

- Restrict to functors with at most one operation per arity.
- There should be one ‘full’ such functor $\top$, with one operation for each possible arity.
- This functor $\top$ should be a monad.
- Selecting basic arities $\Leftrightarrow$ picking a subfunctor $B \subseteq \top$.
- Generating $T \approx \bigcup_n (id \cup B)^n$, the smallest submonad of $\top$ containing $B$. 
Shapely functors: strategy

Find a subcategory $C$ of $\text{Endo}(\text{MGph})$

- stable under composition and
- having a terminal object $\top$.

I.e., such that $\forall C \in C, \exists!$ morphism $C \to \top$.

Indeed:

- $\top$ automatically a monad $\text{via} \ T \circ T \to T$;
- can then generate $\bigcup_n B^n$ amongst subfunctors of $\top$. 
Towards shapely functors I: analytic functors

Subcategory $\text{Analytic}(\text{MGph}) \subseteq \text{Endo}(\text{MGph})$ of functors s.t.

$$T(Y)_n = \sum_{x \in T(1)_n} \text{MGph}(A(x), Y)/G(x)$$

where

- $A(x)$ is the arity of $x$,
- $G(x) \triangleleft \mathcal{S}_{A(x)}$ is a subgroup of the automorphism group of $A(x)$.
- Generalisation of Joyal’s analytic endofunctors on sets.

Miss again!

- Does have a terminal object.
- Not stable under composition.
Towards shapely functors II: cellular functors

Subcategory $\text{Cell}(\text{MGph}) \subseteq \text{Analytic}(\text{MGph}) \subseteq \text{Endo}(\text{MGph})$.

**Miss again!**

- Stable under composition.
- No terminal object!