On the Proof Theory of BCD Intersection Subtyping

CHoCoLa

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February 9, 2017

Teaser / Spoiler

Theorem

BCD subtyping [1983] is a fragment of Lambek calculus [1958].

BCD Intersection Types

BCD Typing System

Terms and types

$$t ::= x \mid \lambda x.t \mid t\,t$$
$$A ::= X \mid A \to A \mid A \cap A \mid \Omega$$

Typing rules

$$\frac{}{\Gamma, x : A \vdash x : A}$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B} \qquad \frac{\Gamma \vdash t : A \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\,u : B}$$

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash t : B}{\Gamma \vdash t : A \cap B} \qquad \frac{}{\Gamma \vdash t : \Omega}$$

$$\frac{\Gamma \vdash t : A \qquad A \leq B}{\Gamma \vdash t : B}$$

Models out of Types

Semantics

$$[\![t]\!] = \{A \mid \vdash t : A\}$$

Subject reduction

$$t \to u \quad \Rightarrow \quad \Gamma \vdash t : A \Rightarrow \Gamma \vdash u : A$$

$$t \to u \quad \Rightarrow \quad [\![t]\!] \subseteq [\![u]\!]$$

Subject expansion

$$t \to u \quad \Rightarrow \quad \Gamma \vdash u : A \Rightarrow \Gamma \vdash t : A$$

$$t \to u \quad \Rightarrow \quad [\![t]\!] \supseteq [\![u]\!]$$

Model

$$t \to u \quad \Rightarrow \quad [\![t]\!] = [\![u]\!]$$

BCD Invariance

Subject β-reduction

$$\bigcap_{i\in I} A_i \to B_i \leq A \to B \quad \overset{(\beta)}{\Rightarrow} \quad \exists J \subseteq I, \quad A \leq \bigcap_{j\in J} A_j \quad \wedge \quad \bigcap_{j\in J} B_j \leq B$$

Subject β-expansion

$$\Gamma \vdash t[^{u}/_{x}] : A \quad \Rightarrow \quad \exists B, \quad \Gamma, x : B \vdash t : A \quad \wedge \quad \Gamma \vdash u : B$$

BCD Subtyping

$$\frac{}{A \leq A} \qquad \frac{A \leq B \qquad B \leq C}{A \leq C}$$

$$\frac{}{A \cap B \leq A} \qquad \frac{}{A \cap B \leq B} \qquad \frac{}{A \leq A \cap A}$$

$$\frac{A \leq C \qquad B \leq D}{A \cap B \leq C \cap D} \qquad \frac{}{A \leq \Omega}$$

$$\frac{C \leq A \qquad B \leq D}{A \to B \leq C \to D}$$

$$\frac{}{(C \to A) \cap (C \to B) \leq C \to (A \cap B)} \qquad \frac{}{\Omega \leq \Omega \to \Omega}$$

Subtyping Revisited

($\cap$,Ω Rules)

$$\frac{}{A \leq A} \qquad \frac{A \leq B \qquad B \leq C}{A \leq C}$$

$$\frac{}{A \cap B \leq A} \qquad \frac{}{A \cap B \leq B} \qquad \frac{}{A \leq A \cap A}$$

$$\frac{A \leq C \qquad B \leq D}{A \cap B \leq C \cap D} \qquad \frac{}{A \leq \Omega}$$

BCD $\cap$,Ω rules = bounded meet-semilattice

Subtyping Revisited

(∩,Ω Rules)

$$\frac{}{A \leq A} \qquad \frac{A \leq B \qquad B \leq C}{A \leq C}$$

$$\frac{A \leq C}{A \cap B \leq C} \qquad \frac{B \leq C}{A \cap B \leq C}$$

$$\frac{C \leq A \qquad C \leq B}{C \leq A \cap B} \qquad \frac{}{A \leq \Omega}$$

Whitman's presentation

Subtyping Revisited

(∩,Ω Rules)

$$\frac{}{A \leq A}$$

$$\frac{A \leq C}{A \cap B \leq C} \qquad \frac{B \leq C}{A \cap B \leq C}$$

$$\frac{C \leq A \qquad C \leq B}{C \leq A \cap B} \qquad \frac{}{A \leq \Omega}$$

Whitman's presentation

Subtyping Revisited

(∩,Ω Rules)

$$\frac{}{A \vdash A}$$

$$\frac{A \vdash C}{A \cap B \vdash C} \qquad \frac{B \vdash C}{A \cap B \vdash C}$$

$$\frac{C \vdash A \qquad C \vdash B}{C \vdash A \cap B} \qquad \frac{}{A \vdash \Omega}$$

&,⊤ additive linear logic

Arrow Rules

Original version

$$\frac{C \leq A \qquad B \leq D}{A \to B \leq C \to D}$$

$$\frac{}{(C \to A) \cap (C \to B) \leq C \to (A \cap B)} \qquad \frac{}{\Omega \leq \Omega \to \Omega}$$

Arrow seen as implication

$$\dfrac{\dfrac{\dfrac{\dfrac{}{C \vdash C} \quad \dfrac{}{A \vdash A}}{C \to A, C \vdash A}}{(C \to A) \cap (C \to B), C \vdash A} \quad \dfrac{\dfrac{\dfrac{}{C \vdash C} \quad \dfrac{}{B \vdash B}}{C \to B, C \vdash B}}{(C \to A) \cap (C \to B), C \vdash B}}{\dfrac{(C \to A) \cap (C \to B), C \vdash A \cap B}{(C \to A) \cap (C \to B) \vdash C \to (A \cap B)}}$$

Remark: more structure in sequents $\Rightarrow$ deeper inference

Sequent Calculus for Subtyping

Pitfalls

$$A \to B \to C \vdash B \to A \to C$$

$$\frac{A \vdash B}{\_ \vdash A \to B}$$

$$\frac{\dfrac{\dfrac{C, D \vdash A \qquad B \vdash E}{A \to B, C, D \vdash E}}{A \to B, C \vdash D \to E}}{A \to B \vdash C \to D \to E}$$

Sequents

$$A \mid B_1, \ldots, B_n \vdash B$$

$$[\![A \mid B_1, \ldots, B_n \vdash B]\!] = A \leq B_1 \to \cdots \to B_n \to B$$

Rules

$$\frac{C \mid \Gamma, A \vdash B}{C \mid \Gamma \vdash A \to B}$$

$$\frac{C \mid \_ \vdash A \qquad B \mid \Gamma \vdash D}{A \to B \mid C, \Gamma \vdash D}$$

$$\frac{}{C \mid \Gamma \vdash \Omega}$$

Sequent Calculus IS

Rules

$$\frac{}{A \mid \vdash A} \qquad \frac{}{C \mid \Gamma \vdash \Omega}$$

$$\frac{C \mid \Gamma \vdash A \qquad C \mid \Gamma \vdash B}{C \mid \Gamma \vdash A \cap B} \qquad \frac{A \mid \Gamma \vdash C}{A \cap B \mid \Gamma \vdash C} \qquad \frac{B \mid \Gamma \vdash C}{A \cap B \mid \Gamma \vdash C}$$

$$\frac{C \mid \Gamma, A \vdash B}{C \mid \Gamma \vdash A \to B} \qquad \frac{C \mid \vdash A \qquad B \mid \Gamma \vdash D}{A \to B \mid C, \Gamma \vdash D}$$

Admissible cuts

$$\frac{A \mid \Gamma \vdash B \qquad B \mid \Delta \vdash C}{A \mid \Gamma, \Delta \vdash C} \qquad \frac{A \mid \vdash B \qquad C \mid \Gamma, B, \Delta \vdash D}{C \mid \Gamma, A, \Delta \vdash D}$$

Properties

Equivalence with BCD

$$A \leq B \qquad \Longleftrightarrow \qquad A \mid \vdash B$$

Subformula property

$$\dfrac{\dfrac{\dfrac{\dfrac{}{C \mid \vdash C} \quad \dfrac{}{A \mid \vdash A}}{C \to A \mid C \vdash A}}{(C \to A) \cap (C \to B) \mid C \vdash A} \quad \dfrac{\dfrac{\dfrac{}{C \mid \vdash C} \quad \dfrac{}{B \mid \vdash B}}{C \to B \mid C \vdash B}}{(C \to A) \cap (C \to B) \mid C \vdash B}}{\dfrac{(C \to A) \cap (C \to B) \mid C \vdash A \cap B}{(C \to A) \cap (C \to B) \mid \vdash C \to (A \cap B)}}$$

Reversible right rules

$$\frac{}{C \mid \Gamma \vdash \Omega} \qquad \frac{C \mid \Gamma \vdash A \qquad C \mid \Gamma \vdash B}{C \mid \Gamma \vdash A \cap B} \qquad \frac{C \mid \Gamma, A \vdash B}{C \mid \Gamma \vdash A \to B}$$

Condition (β) [slightly generalized form]

$$\bigcap_{i \in I} A_i \to B_i \mid A, \Gamma \vdash B \quad \Rightarrow \quad \exists J \subseteq I, \quad A \mid \vdash \bigcap_{j \in J} A_j \quad \wedge \quad \bigcap_{j \in J} B_j \mid \Gamma \vdash B$$

Lambek Calculus

Lambek Calculus L^*

Plagiarism by Anticipation:
L^* is Noncommutative Intuitionistic Multiplicative Linear Logic

Formulas

$$A ::= X \mid A/A \mid A\backslash A \mid A \bullet A$$

Rules

$$\frac{}{A \vdash A}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash B/A} \qquad \frac{\Gamma \vdash A \qquad \Delta, B, \Sigma \vdash C}{\Delta, B/A, \Gamma, \Sigma \vdash C}$$

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A\backslash B} \qquad \frac{\Gamma \vdash A \qquad \Delta, B, \Sigma \vdash C}{\Delta, \Gamma, A\backslash B, \Sigma \vdash C}$$

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \bullet B} \qquad \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, A \bullet B, \Delta \vdash C}$$

Lambek Calculus $\mathbf{L}^*$

Formulas

$$A ::= X \mid A/A \mid A \times A \mid \mathtt{I}$$

Rules

$$\frac{}{A \vdash A}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash B/A} \qquad \frac{\Gamma \vdash A \qquad \Delta, B, \Sigma \vdash C}{\Delta, B/A, \Gamma, \Sigma \vdash C}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \times B} \qquad \frac{\Gamma, A, \Delta \vdash C}{\Gamma, A \times B, \Delta \vdash C} \qquad \frac{\Gamma, B, \Delta \vdash C}{\Gamma, A \times B, \Delta \vdash C}$$

$$\frac{}{\Gamma \vdash \mathtt{I}}$$

A Naive Translation

$$
\begin{aligned}
X^\circ &= X \\
(A \to B)^\circ &= B^\circ / A^\circ \\
(A \cap B)^\circ &= A^\circ \times B^\circ \\
\Omega^\circ &= \mathtt{I}
\end{aligned}
$$

$$
\begin{array}{lcl}
A \mid \Gamma \vdash_{\mathsf{IS}} B & \Rightarrow & A^\circ, \Gamma^\circ \vdash_{\mathsf{L}^*} B^\circ \\
A \mid \Gamma \vdash_{\mathsf{IS}} B & \not\Leftarrow & A^\circ, \Gamma^\circ \vdash_{\mathsf{L}^*} B^\circ
\end{array}
$$

$$
\dfrac{\dfrac{X \to Y \mid \vdash_{\mathsf{IS}} Y \qquad Z \mid X \vdash_{\mathsf{IS}} Z}{Y \to Z \mid X \to Y, X \vdash_{\mathsf{IS}} Z}}{Y \to Z \mid X \to Y \vdash_{\mathsf{IS}} X \to Z}
$$

$$
\dfrac{\dfrac{\dfrac{\overline{X \vdash_{\mathsf{L}^*} X} \qquad \overline{Y \vdash_{\mathsf{L}^*} Y}}{Y/X, X \vdash_{\mathsf{L}^*} Y} \qquad \overline{Z \vdash_{\mathsf{L}^*} Z}}{Z/Y, Y/X, X \vdash_{\mathsf{L}^*} Z}}{Z/Y, Y/X \vdash_{\mathsf{L}^*} Z/X}
$$

A Girard Style Translation

$$X^\bullet = X$$
$$(A \to B)^\bullet = B^\bullet / \neg\neg A^\bullet$$
$$(A \cap B)^\bullet = A^\bullet \times B^\bullet$$
$$\Omega^\bullet = \mathtt{I}$$

$$A \mid \Gamma \vdash_{\mathsf{IS}} B \qquad \Rightarrow \qquad A^\bullet, \neg\neg\Gamma^\bullet \vdash_{\mathsf{L}^*} B^\bullet$$
$$A \mid \Gamma \vdash_{\mathsf{IS}} B \qquad \Leftarrow \qquad A^\bullet, \neg\neg\Gamma^\bullet \vdash_{\mathsf{L}^*} B^\bullet$$

$$\neg\neg A := \neg_R\neg_S A = R/(S/A)$$

[R and S: fresh/dedicated atoms]

Theorem

$$A \leq B \qquad \Longleftrightarrow \qquad A^\bullet \vdash_{\mathsf{L}^*} B^\bullet$$

Translation Proof Sketch

$\boxed{\Rightarrow}$

$$\dfrac{\dfrac{\dfrac{\dfrac{C^\bullet \vdash A^\bullet \qquad \overline{S \vdash S}}{S/A^\bullet, C^\bullet \vdash S}}{S/A^\bullet \vdash S/C^\bullet} \qquad \overline{R \vdash R}}{R/(S/C^\bullet), S/A^\bullet \vdash R}}{R/(S/C^\bullet) \vdash R/(S/A^\bullet)} \qquad \dfrac{\neg\neg C^\bullet \vdash \neg\neg A^\bullet \qquad B^\bullet, \neg\neg\Gamma^\bullet \vdash D^\bullet}{B^\bullet/\neg\neg A^\bullet, \neg\neg C^\bullet, \neg\neg\Gamma^\bullet \vdash D^\bullet}$$

$\boxed{\Leftarrow}$

Lemma 1: $\neg\neg\Gamma^\bullet, \Delta^\bullet \nvdash S$

Lemma 2: $\neg\neg\Gamma^\bullet, S/A^\bullet \vdash R \quad \Rightarrow \quad \Gamma = [C] \quad \wedge \quad C^\bullet \vdash A^\bullet$

$$\dfrac{\dfrac{\text{Lem 1}}{\neg\neg\Delta_1^\bullet, A^\bullet \nvdash S}}{\neg\neg\Delta_1^\bullet \vdash S/A^\bullet} \qquad \dfrac{\dfrac{\dfrac{C^\bullet \vdash A^\bullet}{\neg\neg\Gamma_1^\bullet, S/A^\bullet \vdash R}\ \text{Lem 2}}{\neg\neg\Gamma_1^\bullet \vdash \neg\neg A^\bullet} \qquad B^\bullet, \neg\neg\Gamma_2^\bullet \vdash D^\bullet}{B^\bullet/\neg\neg A^\bullet, \neg\neg\Gamma^\bullet \vdash D^\bullet}$$

$$C^\bullet, \neg\neg\Gamma^\bullet, \neg\neg A^\bullet, \neg\neg\Delta^\bullet \vdash D^\bullet$$

On the Complexity of Subtyping

Reversed Sequent Calculus $\mathsf{IS}_{\mathsf{rev}}$

$$\frac{}{X \mid \ \vdash X} \qquad \frac{}{C \mid \Gamma \vdash \Omega}$$

$$\frac{C \mid \Gamma \vdash A \qquad C \mid \Gamma \vdash B}{C \mid \Gamma \vdash A \cap B} \qquad \frac{A \mid \Gamma \vdash X}{A \cap B \mid \Gamma \vdash X} \qquad \frac{B \mid \Gamma \vdash X}{A \cap B \mid \Gamma \vdash X}$$

$$\frac{C \mid \Gamma, A \vdash B}{C \mid \Gamma \vdash A \to B} \qquad \frac{C \mid \ \vdash A \qquad B \mid \Gamma \vdash X}{A \to B \mid C, \Gamma \vdash X}$$

Property

$$A \mid \Gamma \vdash_{\mathsf{IS}_{\mathsf{rev}}} B \qquad \Longleftrightarrow \qquad A \mid \Gamma \vdash_{\mathsf{IS}} B$$

Proof Search in $\mathsf{IS}_{\mathsf{rev}}$

1. Look at rightmost formula:
   - $C \mid \Gamma \vdash \Omega \quad \Rightarrow \quad$ done
   - $C \mid \Gamma \vdash A \to B \quad \Rightarrow \quad C \mid \Gamma, A \vdash B$
   - $C \mid \Gamma \vdash A \cap B \quad \Rightarrow \quad C \mid \Gamma \vdash A \quad$ and $\quad C \mid \Gamma \vdash B$
   - $C \mid \Gamma \vdash X \quad \mapsto \quad$ go to next step
2. Look at leftmost formula:
   - $Y \mid C, \Gamma \vdash X \quad \Rightarrow \quad$ not provable
   - $Y \mid \vdash X \quad (Y \neq X) \quad \Rightarrow \quad$ not provable
   - $X \mid \vdash X \quad \Rightarrow \quad$ done
   - $\Omega \mid \Gamma \vdash X \quad \Rightarrow \quad$ not provable
   - $A \to B \mid \vdash X \quad \Rightarrow \quad$ not provable
   - $A \to B \mid C, \Gamma \vdash X \quad \Rightarrow \quad C \mid \vdash A \quad$ and $\quad B \mid \Gamma \vdash X$
   - $A \cap B \mid \Gamma \vdash X \quad \Rightarrow \quad A \mid \Gamma \vdash X \quad$ or $\quad B \mid \Gamma \vdash X$

Upper Bound

- Subsequent coordinates

  two pointers to subformulas: $A \mid \Gamma \vdash B$

  - one on each side of $\mid$
  - $\to$-position(A) $\mathtt{R}$-suffix of $\to$-position(B)

- Search space

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X' \mid \vdash X} \quad \dfrac{}{X \mid \vdash X}}{X' \cap X \mid \vdash X} \quad \dfrac{}{Y \mid \vdash Y}}{X \to Y \mid X' \cap X \vdash Y} \quad \dfrac{}{Z \mid X' \cap X \vdash Y}}{(X \to Y) \cap Z \mid X' \cap X \vdash Y}}{Y \to ((X \to Y) \cap Z) \mid Y, X' \cap X \vdash Y} \quad \dfrac{\dfrac{}{Y \mid \vdash Y} \quad \dfrac{\dfrac{}{X \to Y \mid \vdash Z} \quad \dfrac{}{Z \mid \vdash Z}}{(X \to Y) \cap Z \mid \vdash Z}}{Y \to ((X \to Y) \cap Z) \mid Y \vdash Z}}{Y \to ((X \to Y) \cap Z) \mid Y \vdash ((X' \cap X) \to Y) \cap Z}$$

$$\dfrac{Y \to ((X \to Y) \cap Z) \mid Y \vdash ((X' \cap X) \to Y) \cap Z}{Y \to ((X \to Y) \cap Z) \mid \vdash Y \to ((((X' \cap X) \to Y) \cap Z)}$$

Proposition

BCD subtyping is in $\mathtt{ALOGSPACE} = \mathtt{PTIME}$.

Upper Bound

- Subsequent coordinates

  two pointers to subformulas: $A \mid \Gamma \vdash B$
  - one on each side of $\mid$
  - $\to$-position(A) R-suffix of $\to$-position(B)

- Search space

$$\dfrac{\dfrac{\dfrac{\dfrac{}{X' \mid \vdash X} \quad \dfrac{}{X \mid \vdash X}}{X' \cap X \mid \vdash X} \quad \dfrac{}{Y \mid \vdash Y}}{X \to Y \mid X' \cap X \vdash Y} \quad \dfrac{}{Z \mid X' \cap X \vdash Y}}{(X \to Y) \cap Z \mid X' \cap X \vdash Y}$$

$$\dfrac{\dfrac{}{Y \mid \vdash Y} \quad \dfrac{\dfrac{}{X \to Y \mid \vdash Z} \quad \dfrac{}{Z \mid \vdash Z}}{(X \to Y) \cap Z \mid \vdash Z}}{Y \to ((X \to Y) \cap Z) \mid Y \vdash Z}$$

$$\dfrac{\dfrac{\dfrac{(X \to Y) \cap Z) \mid Y, X' \cap X \vdash Y}{Y \to ((X \to Y) \cap Z) \mid Y \vdash (X' \cap X) \to Y} \quad Y \to ((X \to Y) \cap Z) \mid Y \vdash Z}{Y \to ((X \to Y) \cap Z) \mid Y \vdash ((X' \cap X) \to Y) \cap Z}$$

$$\dfrac{Y \to ((X \to Y) \cap Z) \mid Y \vdash ((X' \cap X) \to Y) \cap Z}{Y \to ((X \to Y) \cap Z) \mid \vdash Y \to ((((X' \cap X) \to Y) \cap Z)}$$

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Upper Bound

- Subsequent coordinates

  two pointers to subformulas: $A \mid \Gamma \vdash B$
  - one on each side of $\mid$
  - $\to$-position(A) R-suffix of $\to$-position(B)

- Search space

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X' \mid \vdash X} \quad \dfrac{}{X \mid \vdash X}}{X' \cap X \mid \vdash X} \quad \dfrac{}{Y \mid \vdash Y}}{X \to Y \mid X' \cap X \vdash Y} \quad \dfrac{}{Z \mid X' \cap X \vdash Y}}{(X \to Y) \cap Z \mid X' \cap X \vdash Y} \quad \dfrac{}{Y \mid \vdash Y}}{Y \to ((X \to Y) \cap Z) \mid Y, X' \cap X \vdash Y}$$

$$\dfrac{Y \to ((X \to Y) \cap Z) \mid Y \vdash (X' \cap X) \to Y \qquad \dfrac{\dfrac{}{Y \mid \vdash Y} \quad \dfrac{\dfrac{}{X \to Y \mid \vdash Z} \quad \dfrac{}{Z \mid \vdash Z}}{(X \to Y) \cap Z \mid \vdash Z}}{Y \to ((X \to Y) \cap Z) \mid Y \vdash Z}}{Y \to ((X \to Y) \cap Z) \mid Y \vdash ((X' \cap X) \to Y) \cap Z}$$

$$\dfrac{Y \to ((X \to Y) \cap Z) \mid Y \vdash ((X' \cap X) \to Y) \cap Z}{Y \to ((X \to Y) \cap Z) \mid \vdash Y \to ((((X' \cap X) \to Y) \cap Z)}$$

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$$\dfrac{Y \to ((X \to Y) \cap Z) \mid Y, X' \cap X \vdash Y}{Y \to ((X \to Y) \cap Z) \mid Y \vdash (X' \cap X) \to Y}$$

$$\dfrac{\dfrac{\dfrac{}{Y \mid \vdash Y} \quad \dfrac{\dfrac{}{X \to Y \mid \vdash Z} \quad \dfrac{}{Z \mid \vdash Z}}{(X \to Y) \cap Z \mid \vdash Z}}{Y \to ((X \to Y) \cap Z) \mid Y \vdash Z}}{Y \to ((X \to Y) \cap Z) \mid Y \vdash ((X' \cap X) \to Y) \cap Z}$$

$$\dfrac{Y \to ((X \to Y) \cap Z) \mid Y \vdash ((X' \cap X) \to Y) \cap Z}{Y \to ((X \to Y) \cap Z) \mid \vdash Y \to ((((X' \cap X) \to Y) \cap Z)}$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X' \mid \vdash X} \quad \dfrac{}{X \mid \vdash X}}{X' \cap X \mid \vdash X} \quad \dfrac{}{Y \mid \vdash Y}}{X \to Y \mid X' \cap X \vdash Y} \quad \dfrac{}{Z \mid X' \cap X \vdash Y}}{(X \to Y) \cap Z \mid X' \cap X \vdash Y} \quad \dfrac{}{Y \mid \vdash Y}}{Y \to ((X \to Y) \cap Z) \mid Y, X' \cap X \vdash Y}$$

$$\dfrac{Y \to ((X \to Y) \cap Z) \mid Y, X' \cap X \vdash Y}{Y \to ((X \to Y) \cap Z) \mid Y \vdash (X' \cap X) \to Y} \quad \dfrac{\dfrac{}{Y \mid \vdash Y} \quad \dfrac{\dfrac{}{X \to Y \mid \vdash Z} \quad \dfrac{}{Z \mid \vdash Z}}{(X \to Y) \cap Z \mid \vdash Z}}{Y \to ((X \to Y) \cap Z) \mid Y \vdash Z}$$

$$\dfrac{Y \to ((X \to Y) \cap Z) \mid Y \vdash (X' \cap X) \to Y \quad Y \to ((X \to Y) \cap Z) \mid Y \vdash Z}{Y \to ((X \to Y) \cap Z) \mid Y \vdash ((X' \cap X) \to Y) \cap Z}$$

$$\dfrac{Y \to ((X \to Y) \cap Z) \mid Y \vdash ((X' \cap X) \to Y) \cap Z}{Y \to ((X \to Y) \cap Z) \mid \vdash Y \to ((((X' \cap X) \to Y) \cap Z)}$$

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  - one on each side of $\mid$
  - $\to$-position(A) R-suffix of $\to$-position(B)

- Search space

$$X' \mid \; \vdash X \qquad X \mid \; \vdash X$$
$$X' \cap X \mid \; \vdash X \qquad Y \mid \; \vdash Y$$
$$X \to Y \mid X' \cap X \vdash Y \qquad Z \mid X' \cap X \vdash Y$$
$$Y \mid \; \vdash Y \qquad (X \to Y) \cap Z \mid X' \cap X \vdash Y$$
$$Y \to ((X \to Y) \cap Z) \mid Y, X' \cap X \vdash Y$$
$$Y \to ((X \to Y) \cap Z) \mid Y \vdash (X' \cap X) \to Y$$
$$X \to Y \mid \; \vdash Z \qquad Z \mid \; \vdash Z$$
$$Y \mid \; \vdash Y \qquad (X \to Y) \cap Z \mid \; \vdash Z$$
$$Y \to ((X \to Y) \cap Z) \mid Y \vdash Z$$
$$Y \to ((X \to Y) \cap Z) \mid Y \vdash ((X' \cap X) \to Y) \cap Z$$
$$Y \to ((X \to Y) \cap Z) \mid \; \vdash Y \to ((((X' \cap X) \to Y) \cap Z)$$

Proposition

BCD subtyping is in ALOGSPACE = PTIME.

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Variations and Extensions

BCD with Constructors

Covariant constructors in BCD

$$A ::= \kappa(\vec{A}) \mid A \to A \mid A \cap A \mid \Omega$$

$$\frac{\kappa \preccurlyeq \kappa'}{\kappa(\vec{A}) \leq \kappa'(\vec{A})} \qquad \frac{\cdots\ A_i \leq B_i\ \cdots}{\kappa(\vec{A}) \leq \kappa(\vec{B})} \qquad \frac{}{\kappa(\vec{A}) \cap \kappa(\vec{B}) \leq \kappa(\vec{A} \cap \vec{B})}$$

From implication-centric to intersection-centric sequents

$$[\![A \mid B_1, \ldots, B_n \vdash B]\!] = A \leq B_1 \to \cdots \to B_n \to B$$

$$[\![C_1, \ldots, C_n \vdash D]\!] = C_1 \cap \cdots \cap C_n \leq D$$

$$\dfrac{\dfrac{\dfrac{}{C \vdash C} \quad \dfrac{}{C \vdash C} \quad \dfrac{\dfrac{\dfrac{}{A \vdash A}}{A, B \vdash A} \quad \dfrac{\dfrac{}{B \vdash B}}{A, B \vdash B}}{A, B \vdash A \cap B}}{C \to A, C \to B \vdash C \to (A \cap B)}}{(C \to A) \cap (C \to B) \vdash C \to (A \cap B)}$$

Sequent Calculus for General Constructors

Formulas

$$A ::= \kappa(\vec{A};\vec{A}) \mid A \cap A$$

Rules (sketch)

$$\frac{\Gamma \vdash C}{\sigma\Gamma \vdash C} \qquad \frac{\Gamma \vdash C}{\Gamma, \kappa(\vec{A}) \vdash C}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \cap B} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \cap B \vdash C}$$

$$\frac{\begin{array}{ccc}\forall i & & \forall i \\ \kappa_i \preccurlyeq \kappa & A \vdash A_i & B_1, \ldots, B_n \vdash B\end{array}}{\kappa_1(A_1;B_1), \ldots, \kappa_n(A_n;B_n) \vdash \kappa(A;B)}$$

Admissible rules

$$\frac{\Gamma \vdash C}{\Gamma, A \vdash C} \qquad \frac{}{A \vdash A} \qquad \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C} \qquad \frac{\Gamma \vdash A \qquad A, \Delta \vdash C}{\Gamma, \Delta \vdash C}$$

η-Invariance

Necessary and sufficient condition

$$\bigcap_{i \in I} A_i \to B_i \;\leq\; X \;\leq\; \bigcap_{i \in I} A_i \to B_i$$

Scott solution

$X \equiv \Omega \to X$

$$\frac{\mid \vdash A \qquad B \mid \vdash X}{A \to B \mid \Gamma \vdash X} \qquad\qquad \frac{}{X \mid \Gamma \vdash X}$$

Generic approach

$X \equiv \bigcap_{i \in I} A_i^X \to B_i^X$

$$\frac{\cdots \quad C \mid \Gamma, A_i^X \vdash B_i^X \quad \cdots}{C \mid \Gamma \vdash X} \qquad\qquad \frac{A \mid \vdash A_i^X \qquad B_i^X \mid \Gamma \vdash C}{X \mid A, \Gamma \vdash C}$$