Up-to techniques for behavioural metrics

via fibrations

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joint work with Filippo Bonchi and Barbara König
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FONDAMENTALE

Overview

In this talk

Part I

- Up-to techniques for behavioural equivalences
- Behavioural metrics coinductively
- A running example: computing distances between regular languages more efficiently

Part II

- A generic framework for proving soundness of up-to techniques using liftings of functors
- The Wasserstein lifting of a Set-functor
- Application to the running example

Part I: Introducing up-to techniques for behavioural metrics

Up-to techniques for behavioural equivalences

Used for proving behavioural equivalences of processes in concurrency theory:

[Milner. 1989]
Communication and Concurrency. Prentice Hall.

[Pous and Sangiorgi. 2011]
Enhancements of the coinductive proof method. In Advanced Topics in Bisimulation and Coinduction. Cambridge University Press

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Applications for automata: The HKC algorithm for checking language equivalence for NFAs

[Bonchi and Pous. 2013]
Checking NFA equivalence with bisimulations up to congruence. In POPL. ACM, 457–468.

So what are up-to techniques?

In many cases behavioural equivalences are coinductive predicates, i.e., they can be expressed as the greatest fixpoint νb of a monotone function

$$b\colon \mathrm{Rel}_Q \to \mathrm{Rel}_Q\,,$$

where Rel_Q is the complete lattice of relations on the state space Q.

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Sound and Compatible up-to techniques

Definition (Sound up-to technique)
A function $f\colon \mathsf{Rel}_Q \to \mathsf{Rel}_Q$ is sound w.r.t. νb when the coinduction up-to f proof principle is valid:

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Lemma: Compatibility implies soundness.

Example: language equivalence for NFAs

Example

Up-to congruence closure is a sound up-to technique w.r.t. language equivalence for determinized NFAs.

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- Language equivalence coincides with bisimilarity for deterministic automata, i.e. is νb for $b \colon \mathrm{Rel}_Q \to \mathrm{Rel}_Q$ given by

$$b(r) = \{(x, y) \mid o(x) = o(y) \text{ and } \forall a \in A, (\delta_a(x), \delta_b(y)) \in r\}$$

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- The determinization of an NFA with states Q also has an algebraic structure: $\mathcal{P}Q$ is a join-semilattice, and, moreover:

$$L(X) \cup L(Y) = L(X \cup Y), \qquad \text{for } X, Y \in \mathcal{P}Q\,.$$

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Define up-to congruence as the map $\mathrm{cgr}\colon \mathrm{Rel}_{\mathcal{P}Q} \to \mathrm{Rel}_{\mathcal{P}Q}$ sending a relation r to its closure under equivalence and the rule:

$$\frac{(X_1, Y_1) \in r \quad (X_2, Y_2) \in r}{(X_1 \cup X_2, Y_1 \cup Y_2) \in r}$$

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The proof principle:

$$\frac{(x, y) \in r \qquad r \subseteq b(\mathrm{cgr}(r))}{(x, y) \in \nu b}$$

is valid. The HKC algorithm [Bonchi and Pous, POPL'13] computes on-the-fly a bisimulation up-to congruence relating $\{x\}$ and $\{y\}$.

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We have seen how one can lift distances between states of a system to distances between probability distributions on these state spaces.

Computing behavioural metrics is not easy... Can we use up-to techniques? In the process we also discuss systematic liftings of arbitrary Set-functors to pseudo-metrics.

Running example: distance between regular languages

Definition (Shortest distinguishing word distance)
Given two languages L and K, define

$$d_{sdw}(L, K) = c^{|w|} ,$$

where c is a constant such that $0 < c < 1$ and w is the shortest word which belongs to exactly one of the languages L, K.

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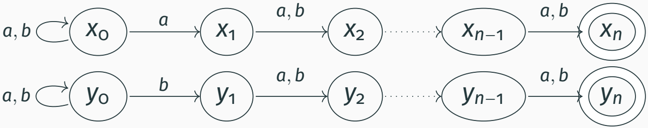
Example: In the NFA below $d_{sdw}(x_0, y_0) \leq c^n$.



How can we prove such inequalities more efficiently?

Running example: distance between regular languages

For a deterministic automaton, the distance d_{sdw} between the languages accepted by two states can be expressed as the greatest fixpoint νb of a function

$$b\colon [0,1]^{Q\times Q} \to [0,1]^{Q\times Q}$$

defined on the complete lattice $[0,1]^{Q\times Q}$ ordered with the reversed point-wise order $\leq$:

$$b(d)(q_1,q_2) = \begin{cases} 1, & \text{if only one of } q_1, q_2 \\ & \text{is accepting} \\ c \cdot \max\limits_{a\in A} d(\delta_a(q_1), \delta_a(q_2)), & \text{otherwise} \end{cases}$$

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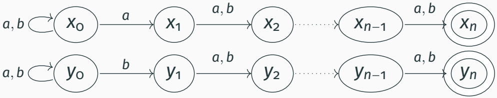

Option 1 (coinduction): Determinize the NFA and find a distance $\overline{d}$ such that $\overline{d}(\{x_0\},\{y_0\}) \leq c^n$ and $\overline{d} \prec b(\overline{d})$. Use the coinduction principle:

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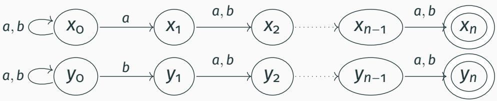

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We obtain $\overline{d} < d_{sdw}$, and hence $d_{sdw}(\{x_0\}, \{y_0\}) \leq c^n$.

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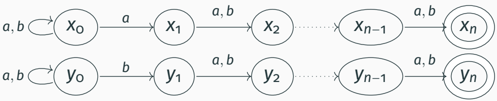

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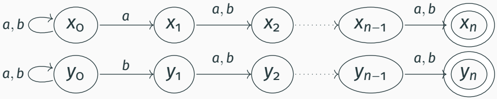
We obtain $\overline{d} \prec d_{sdw}$, and hence $d_{sdw}(\{x_0\},\{y_0\}) \leq c^n$.

Disadvantage: we need to compute $\overline{d}$ for exponentially many pairs of states.

Running example: distance between regular languages



Option 2 (coinduction up-to): use a sound up-to context technique which closes a $[0,1]$-valued relation under the rules:

$$\frac{d(X_1,X_2) \le r}{f(d)(X_1,X_2) \le r} \qquad \frac{f(d)(X_1,X_2) \le r \quad f(d)(Y_1,Y_2) \le r}{f(d)(X_1 \cup Y_1, X_2 \cup Y_2) \le r}$$

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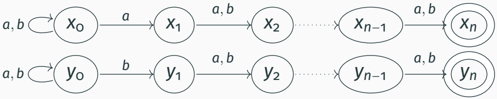

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Find a relaxed invariant $\overline{d}$ such that $\overline{d} \prec b(f(\overline{d}))$ and $\overline{d}(\{x_0\}, \{y_0\}) \le c^n$. Use the coinduction up-to principle to conclude $\overline{d} \prec d_{sdw}$, and hence $d_{sdw}(x_0, y_0) \le c^n$.

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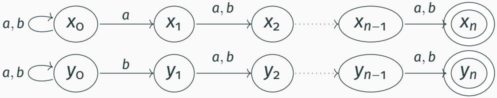

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Define $\overline{d}(\{x_i\}, \{y_j\}) = c^{n-\max\{i,j\}}$ and $\overline{d}(X, Y) = 1$ for all other values.

Notice that it suffices to define $\overline{d}$ on a linear number of pairs.

Running example: distance between regular languages

From the generic framework developed in the rest of the talk, we will establish:

- how both b and f can be expressed in terms of so called Wasserstein liftings of functors to $[0, 1]$-valued relations.
- why the soundness of f follows from a generic framework developed previously for up-to techniques in a fibrational setting.

Part II: Soundness of up-to techniques for behavioural metrics

In the rest of the talk

Use the fibrational framework of our previous CSL-LICS'2014 paper to prove the soundness of the quantitative version of the up-to congruence technique. To this end:

- Coinductive predicates (in particular, behavioural metrics) can be expressed via functor liftings
- Up-to techniques can also be expressed via functor liftings
- Discuss functor liftings and in particular what we will call the Wasserstein lifting of a functor
- Apply all this machinery in the example of d_{sdw}

Coinductive predicates via functor liftings

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Coinductive predicates describing properties of a coalgebra $\xi\colon X \to FX$ can be seen as post-fixpoints of a composite map b

$$b\colon \mathsf{Rel}_X \xrightarrow{\overline{F}} \mathsf{Rel}_{FX} \xrightarrow{\xi^{-1}} \mathsf{Rel}_X$$

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$$b\colon \mathsf{Rel}_X \xrightarrow{\overline{F}} \mathsf{Rel}_{FX} \xrightarrow{\xi^{-1}} \mathsf{Rel}_X$$

where $\overline{F}$ is a "lifting" of F mapping relations on X to relations on FX and for $R \subseteq FX \times FX$

$$(x, y) \in \xi^{-1}(R) \text{ iff } (\xi(x), \xi(y)) \in R$$

Example 1: language equivalence via functor liftings

Forgetting about the initial state, a DFA is a coalgebra for the functor $FX = 2 \times X^A$, i.e. a map of the form

$$\langle o, \delta \rangle : X \to 2 \times X^A ,$$

with $o(q) = 1$ iff q is accepting and $\delta(q)(a) = \delta_a(q)$.

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where $\overline{F}$ denotes here the so-called canonical lifting of F, i.e., for $R \subseteq X \times X$ and $(o_i, \phi_i) \in FX$ we have

$$(o_1, \phi_1)\ \overline{F}(R)\ (o_2, \phi_2) \text{ iff } \begin{cases} o_1 = o_2 \\ \forall a \in A \quad \phi_1(a)\ R\ \phi_2(a) \end{cases}$$

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The distance d_{sdw} is the largest fixpoint of the monotone map b on the lattice of $[0,1]$-valued relations, ordered by $\prec$ (the point-wise reverse order on the reals). This is obtained as the composite

$$b : [0,1]\text{-Rel}_X \xrightarrow{\overline{F}} [0,1]\text{-Rel}_{FX} \xrightarrow{\langle o,\delta\rangle^{-1}} [0,1]\text{-Rel}_X$$

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The distance d_{sdw} is the largest fixpoint of the monotone map b on the lattice of $[0,1]$-valued relations, ordered by $<$ (the point-wise reverse order on the reals). This is obtained as the composite

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where $\overline{F}$ is defined for $d : X \times X \to [0,1]$ by

$$(o_1, \phi_1)\ \overline{F}(d)\ (o_2, \phi_2) \text{ iff } \begin{cases} 1 & \text{if } o_1 \neq o_2 \\ c \cdot \max\limits_{a \in A} d(\phi_1(a), \phi_2(a)) & \text{otherwise} \end{cases}$$

Up-to context closure via functor liftings

If we consider now a system, which is not only modelled as a coalgebra, but is also equipped with a compatible algebraic structure, it makes sense to consider the up-to congruence technique with respect to this algebraic structure.

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Definition (Bialgebra)
Consider two functors F, T and a natural transformation $\zeta \colon TF \Rightarrow FT$. A bialgebra for ζ is a tuple (X, α, ξ) such that

$$\alpha \colon TX \to X \text{ is a } T\text{-algebra}, \qquad \xi \colon X \to FX \text{ is an } F\text{-coalgebra}$$

so that the next diagram commutes.

$$\begin{tikzcd} TX \arrow[r, "\alpha"] \arrow[d, "T\xi"'] & X \arrow[r, "\xi"] & FX \\ TFX \arrow[rr, "\zeta_X"'] & & FTX \arrow[u, "F\alpha"'] \end{tikzcd}$$

Definition (Bialgebra)
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Example
The determinization of an NFA with states Q is a bialgebra of the form $(\mathcal{P}Q, \cup, \gamma)$ for the functors $FX = 2 \times X^A$, $TX = \mathcal{P}X$ and $\zeta_X\colon \mathcal{P}(2 \times X^A) \to 2 \times (\mathcal{P}X)^A$ defined for $M \subseteq 2 \times X^A$ by

$$\zeta_X(M) = \Big(\bigvee_{(b,f)\in M} b, [a \mapsto \{f(a) \mid (b,f) \in M\}]\Big)$$

Example 1: Up-to context closure for determinized NFAs

To sum up, a determinized NFA has both algebra and coalgebra structures, which are related by a distributive law:

$$\cup\colon \mathcal{P}\mathcal{P}Q \to \mathcal{P}Q \qquad \text{and} \qquad \xi\colon \mathcal{P}Q \to 2 \times (\mathcal{P}Q)^A \,.$$

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Context closure $\mathsf{ctx}\colon \mathsf{Rel}_{\mathcal{P}Q} \to \mathsf{Rel}_{\mathcal{P}Q}$ of a relation r on $\mathcal{P}Q$ is defined via the rule:

$$\frac{(X_1, Y_1) \in r \qquad (X_2, Y_2) \in r}{(X_1 \cup X_2, Y_1 \cup Y_2) \in \mathsf{ctx}(r)}$$

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This can be seen as the composite map:

$$\mathsf{ctx}\colon \mathsf{Rel}_{\mathcal{P}Q} \xrightarrow{\overline{\mathcal{P}}} \mathsf{Rel}_{\mathcal{P}\mathcal{P}Q} \xrightarrow{\Sigma_\cup} \mathsf{Rel}_{\mathcal{P}Q}$$

where $\overline{\mathcal{P}}$ is the canonical relation lifting of $\mathcal{P}$ and $\Sigma_\cup$ is forward image along the $\cup$.

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where $\overline{\mathcal{P}}$ is the canonical relation lifting of $\mathcal{P}$ and $\Sigma_{\cup}$ is forward image along the $\cup$, i.e., for $R \in \mathrm{Rel}_{\mathcal{P}Q}$ and $S \in \mathrm{Rel}_{\mathcal{P}\mathcal{P}Q}$:

- $(\mathcal{X}, \mathcal{Y}) \in \overline{\mathcal{P}}(R)$ iff $\begin{cases} \forall A \in \mathcal{X}, \exists B \in \mathcal{Y}\ (A, B) \in R \\ \forall B \in \mathcal{Y}, \exists A \in \mathcal{X}\ (A, B) \in R \end{cases}$
- $(X, Y) \in \Sigma_{\cup}(S)$ iff $X = \cup\mathcal{X}$, $Y = \cup\mathcal{Y}$ and $(\mathcal{X}, \mathcal{Y}) \in S$.

Example 2: Quantitative up-to context closure for determinized NFAs

Definition (Quantitative context closure)
The quantitative context closure $f\colon \mathrm{Rel}_{\mathcal{P}Q} \to \mathrm{Rel}_{\mathcal{P}Q}$ considered in the running example is defined as the composite

$$f\colon [0,1]\text{-}\mathrm{Rel}_{\mathcal{P}Q} \xrightarrow{\overline{\mathcal{P}}} [0,1]\text{-}\mathrm{Rel}_{\mathcal{P}\mathcal{P}Q} \xrightarrow{\Sigma_{\cup}} [0,1]\text{-}\mathrm{Rel}_{\mathcal{P}Q}$$

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- $\overline{\mathcal{P}}(d)(X_1, X_2) = \sup\{\sup_{x_1 \in X_1} \inf_{x_2 \in X_2} d(x_1, x_2), \sup_{x_2 \in X_2} \inf_{x_1 \in X_1} d(x_1, x_2)\}$

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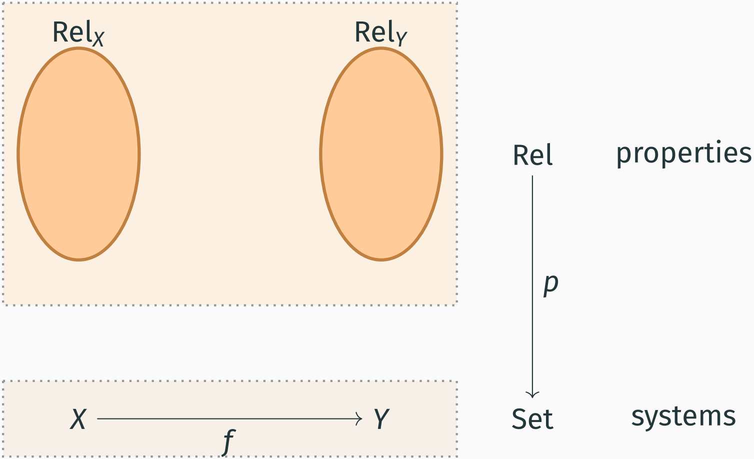
- $\Sigma_\cup(s)(X, Y) = \inf\{s(\mathcal{X}, \mathcal{Y}) \mid \cup\mathcal{X} = X, \cup\mathcal{Y} = Y\}$.

A common framework for quantitative and qualitative setting



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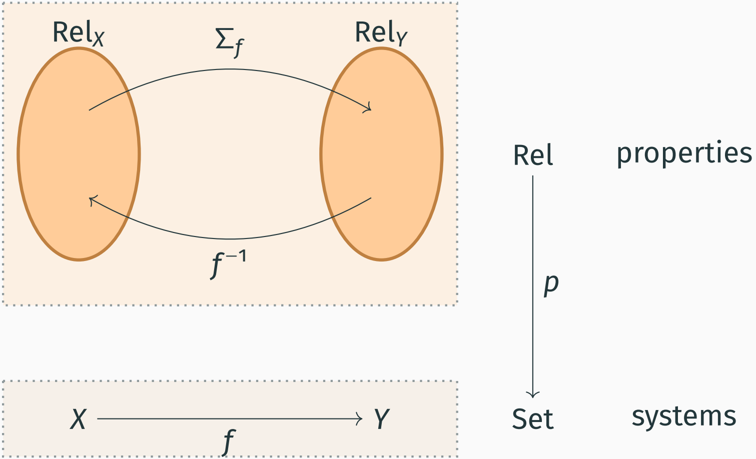

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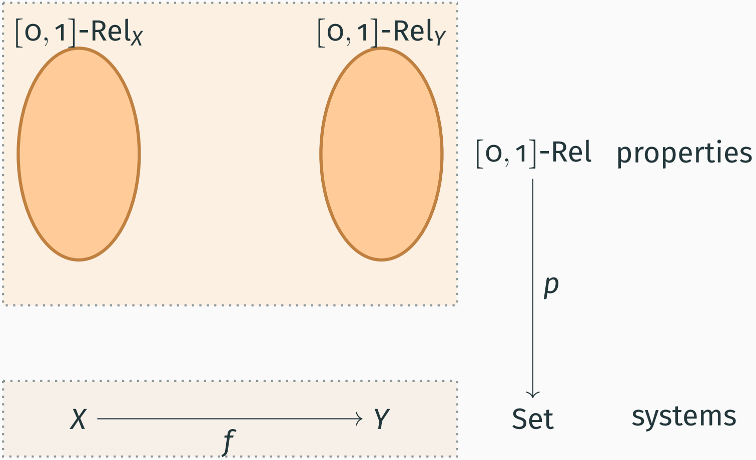

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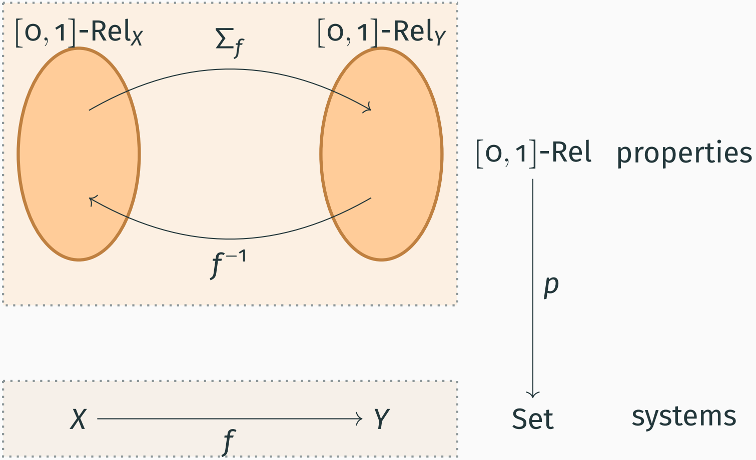

Bifibrations

A functor $p \colon \mathbb{P} \to \mathbb{B}$ is called a fibration when for every morphism $f \colon X \to Y$ in $\mathbb{B}$ and every R in $\mathbb{P}$ with $p(R) = Y$ there exists a map $\widetilde{f}_R \colon f^*(R) \to R$ such that $p(\widetilde{f}_R) = f$ and satisfying the universal property:

$$
\begin{tikzcd}
 & R \\
 & \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

For all maps $g : Z \to X$ in $\mathbb{B}$ and $u \colon Q \to R$ in $\mathcal{P}$ sitting above fg (i.e., $p(u) = fg$) there is a unique map $v \colon Q \to f^*(R)$ such that $u = \widetilde{f}_R v$ and $p(v) = g$.

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$$\begin{tikzcd}
Q \arrow[dr, dotted, "\exists! v"'] \arrow[drr, bend left, "\forall u"] & & \\
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\end{tikzcd}$$

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In a bifibration every reindexing has a left adjoint $\Sigma_f \dashv f^*$.

Liftings of functors

Given a fibration $p\colon \mathbb{P} \to \mathbb{B}$ and a functor $F\colon \mathbb{B} \to \mathbb{B}$, a lifting of F is a functor $\widehat{F}\colon \mathbb{P} \to \mathbb{P}$ such that

$$\begin{tikzcd}
\mathbb{P} \arrow[r, "\widehat{F}"] \arrow[d, "p"'] & \mathbb{P} \arrow[d, "p"] \\
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For every $r \in \mathbb{P}_Y$ and $f\colon X \to Y$ in $\mathbb{B}$, we have a canonical natural transformation

$$\overline{F} \circ f^*(R) \to (Ff)^* \circ \overline{F}(R)\,.$$

The lifting $\widehat{F}$ is called a fibred lifting of F when the above natural transformation is an isomorphism for every r.

A theorem from CSL-LICS'14

Consider a fibration $p\colon \mathbb{P} \to \mathbb{B}$ and a bialgebra in $\mathbb{B}$

$$\begin{tikzcd} TX \arrow[r, "\alpha"] \arrow[d, "T\gamma"'] & X \arrow[r, "\gamma"] & FX \\ TFX \arrow[rr, "\zeta_X"'] & & FTX \arrow[u, "F\alpha"'] \end{tikzcd}$$

Consider two liftings $\overline{T}$ and $\overline{F}$ of T, respectively F, to the category $\mathbb{P}$ such that there exists a lifting $\overline{\zeta}\colon \overline{T} \circ \overline{F} \Rightarrow \overline{F} \circ \overline{T}$ of the distributive law ζ. Then the up-to technique

$$f\colon \mathbb{P}_X \xrightarrow{\overline{T}} \mathbb{P}_{TX} \xrightarrow{\Sigma_\alpha} \mathbb{P}_X$$

is sound with respect to the coinductive predicate defined by

$$b\colon \mathbb{P}_X \xrightarrow{\overline{F}} \mathbb{P}_{FX} \xrightarrow{\xi^*} \mathbb{P}_X$$

Lifting of functors to many-valued relations

Quantale valued predicates and relations

Definition
A quantale $\mathcal{V}$ is a complete lattice equipped with an associative operation $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ which is distributive on both sides over arbitrary joins $\bigvee$. We assume the tensor is commutative and has a unit 1.

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Given a set X and a quantale $\mathcal{V}$, a $\mathcal{V}$-valued predicate on X is a map $p : X \to \mathcal{V}$. A $\mathcal{V}$-valued relation on X is a map $r : X \times X \to \mathcal{V}$.

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Given two $\mathcal{V}$-valued predicates $p, q : X \to \mathcal{V}$, we say that

$$p \le q \iff \forall x \in X.\ p(x) \le q(x).$$

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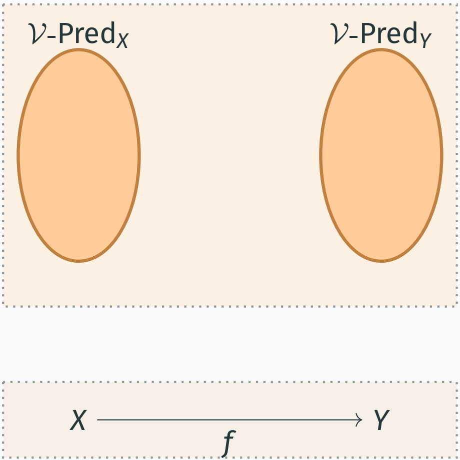
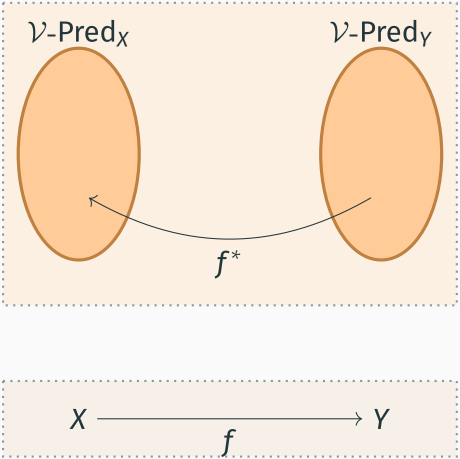
$$p \leq q \iff \forall x \in X.\ p(x) \leq q(x).$$

A morphism between $\mathcal{V}$-valued predicates $p : X \to \mathcal{V}$ and $q : Y \to \mathcal{V}$ is a map $f : X \to Y$ such that $p \leq q \circ f$. We consider the category $\mathcal{V}$-Pred whose objects are $\mathcal{V}$-valued predicates and arrows are as above.



$$\begin{array}{ll} \mathcal{V}\text{-Pred} & \quad\text{properties} \\ \big\downarrow p & \\ \mathsf{Set} & \quad\text{systems} \end{array}$$

The bifibrations of $\mathcal{V}$-valued predicates



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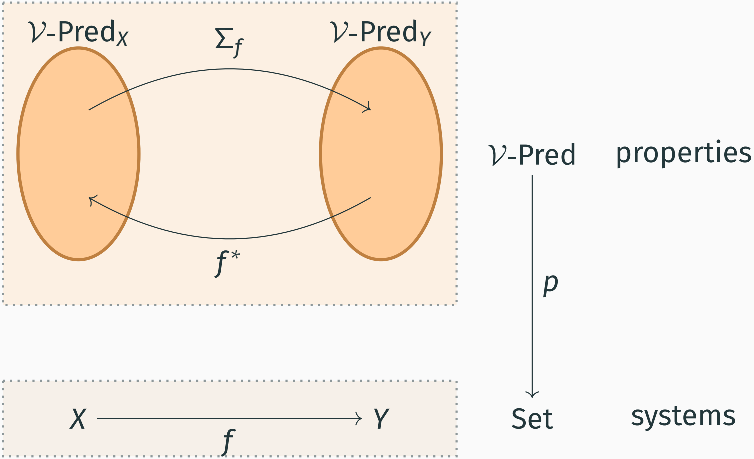

- $f^*(p) = p \circ f$,
- $\Sigma_f(p)(y) = \bigvee \{ p(x) \mid x \in f^{-1}(y) \}$

The bifibrations of $\mathcal{V}$-valued predicates and relations

We have a change-of-base situation, where $\Delta X = X \times X$.

$$\begin{tikzcd}
\mathcal{V}\text{-Rel} \arrow[r, "\iota"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{V}\text{-Pred} \arrow[d] \\
\text{Set} \arrow[r, "\Delta"'] & \text{Set}
\end{tikzcd}$$

We obtain a bifibration $\mathcal{V}\text{-Rel} \to \text{Set}$.

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We obtain a bifibration $\mathcal{V}\text{-Rel} \to \mathsf{Set}$.

Remark: $\mathcal{V}$-categories are $\mathcal{V}$-valued relation $r : X \times X \to \mathcal{V}$ that are

- reflexive if for all $x \in X$ we have $r(x, x) \geq 1$, and
- transitive if for all $x, y, z \in X$ we have $r(y, z) \otimes r(x, y) \leq r(x, y)$.

We also obtain a bifibration $\mathcal{V}\text{-Cat} \to \mathsf{Set}$.

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We also obtain a bifibration $\mathcal{V}\text{-Cat} \to \mathsf{Set}$.

For the quantale $([0,\infty], \geq_{[0,\infty]}, +, 0)$ these are the generalized pseudo-metrics from Lawvere's 1973 seminal paper.

A systematic way of lifting functors

$$\begin{tikzcd}
\mathcal{V}\text{-Rel} \arrow[r, "\iota"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{V}\text{-Pred} \arrow[d] \\
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Step 1: Lift a Set-functor F to a functor $\widehat{F}$ the category of $\mathcal{V}$-predicates.

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Step 3: When does $\overline{F}$ restrict to $\mathcal{V}$-categories?

A systematic way of lifting functors

$$\begin{tikzcd} \mathcal{V}\text{-Rel} \arrow[r, "\iota"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{V}\text{-Pred} \arrow[d] \\ \mathsf{Set} \arrow[r, "\Delta"'] & \mathsf{Set} \end{tikzcd}$$

Step 1: Lift a Set-functor F to a functor $\widehat{F}$ on the category $\mathcal{V}$-Pred.

Proposition. There is a one-to-one correspondence between

- fibred liftings $\overline{F}$ of F to $\mathcal{V}$-Pred,
- monotone natural transformations $\mathcal{V}^{-} \Rightarrow \mathcal{V}^{F-}$,
- monotone evaluation maps $ev : F\mathcal{V} \to \mathcal{V}$.

We also define $ev_{\mathsf{can}} : F\mathcal{V} \to \mathcal{V}$ as follows:

$$ev_{\mathsf{can}}(u) = \bigvee\{r \mid u \in F(\uparrow r)\}.$$

A systematic way of lifting functors

$$\begin{tikzcd}
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\mathrm{Set} \arrow[r, "\Delta"'] & \mathrm{Set}
\end{tikzcd}$$

Step 2: Transfer the predicate lifting $\widehat{F}$ to a lifting $\overline{F}$ on $\mathcal{V}$-relations.

$$\begin{tikzcd}
\mathcal{V}\text{-Rel}_X \arrow[rr, dashed, "\overline{F}_X"] \arrow[d, "\iota_X"'] & & \mathcal{V}\text{-Rel}_{FX} \\
\mathcal{V}\text{-Pred}_{\Delta X} \arrow[r, "\widehat{F}_{\Delta X}"'] & \mathcal{V}\text{-Pred}_{F\Delta X} \arrow[r, "\Sigma_{\lambda_X}"'] & \mathcal{V}\text{-Pred}_{\Delta FX} \arrow[u, "{\iota_{FX}}^{-1}"']
\end{tikzcd}$$

where $\lambda_X \colon F(X \times X) \to FX \times FX$ is given by the pairing of projections $\langle F\pi_1, F\pi_2 \rangle$.

A systematic way of lifting functors

$$\begin{tikzcd}
\mathcal{V}\text{-Rel} \arrow[r, "\iota"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{V}\text{-Pred} \arrow[d] \\
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Step 2: Transfer the predicate lifting $\widehat{F}$ to a lifting $\overline{F}$ on $\mathcal{V}$-relations.

Concretely, the lifting $\overline{F}$ is defined via "couplings":

$$\overline{F}(p)(t_1, t_2) = \bigvee \{\widehat{F}(p)(t) \mid t \in F(X \times X), F\pi_i(t) = t_i\}$$

We call $\overline{F}$ the Wasserstein lifting associated with $\widehat{F}$.

A systematic way of lifting functors

Step 3: When does $\overline{F}$ restrict to $\mathcal{V}$-categories?

We have the following characterization theorem, where κ_X denotes the constant to 1 predicate on X, and for two predicates $p, q \colon X \to \mathcal{V}$ we denote by $p \otimes q \colon X \to \mathcal{V}$ the predicate mapping x to $p(x) \otimes q(x)$.

Theorem. *Assume* $\widehat{F}$ *is a lifting of* F *to* $\mathcal{V}$-Pred *and* $\overline{F}$ *is the corresponding* $\mathcal{V}$-Rel *Wasserstein lifting. Then*

- *If* $\widehat{F}(\kappa_X) \geq \kappa_{FX}$ *then* $\overline{F}$ *preserves reflexive relations,*

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- $\overline{F}$ *preserves symmetric relations.*

A systematic way of lifting functors

Step 3: When does $\overline{F}$ restrict to $\mathcal{V}$-categories?

We have the following characterization theorem, where κ_X denotes the constant to 1 predicate on X, and for two predicates $p, q : X \to \mathcal{V}$ we denote by $p \otimes q : X \to \mathcal{V}$ the predicate mapping x to $p(x) \otimes q(x)$.

Theorem. *Assume $\widehat{F}$ is a lifting of F to $\mathcal{V}$-*Pred *and $\overline{F}$ is the corresponding $\mathcal{V}$-*Rel *Wasserstein lifting. Then*

- *If $\widehat{F}(\kappa_X) \geq \kappa_{FX}$ then $\overline{F}$ preserves reflexive relations,*
- *If $\widehat{F}$ is a fibred lifting, F preserves weak pullbacks and $\widehat{F}(p \otimes q) \geq \widehat{F}(p) \otimes \widehat{F}(q)$ then $\overline{F}$ preserves transitive relations,*
- *$\overline{F}$ preserves symmetric relations.*

Whenever F preserves weak pullbacks the canonical evaluation lifting $\widehat{F}_{can}$ satisfies the above conditions.

Lifting distributive laws to Wasserstein liftings

Theorem
Assume the natural transformation $\zeta\colon T \circ F \Rightarrow F \circ T$ *lifts to a natural transformation* $\widehat{\zeta}\colon \widehat{T} \circ \widehat{F} \Rightarrow \widehat{F} \circ \widehat{T}$ *between* $\mathcal{V}$*-predicate liftings and that we have* $\widehat{T} \circ \Sigma_{\lambda^F_X} \leq \Sigma_{T\lambda^F_X} \circ \widehat{T}$. *Then* ζ *lifts to a distributive law* $\overline{\zeta}\colon \overline{T} \circ \overline{F} \Rightarrow \overline{F} \circ \overline{T}$ *between the corresponding Wasserstein liftings.*

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Theorem
Assume that $\zeta\colon T \circ F \Rightarrow F \circ T$ *is a natural transformation and that, furthermore,* T *preserves weak pullbacks and* F *preserves intersections. Then* ζ *lifts to a natural transformation*

$$\widehat{\zeta}\colon \widehat{T}_{\mathrm{can}} \circ \widehat{F}_{\mathrm{can}} \Rightarrow \widehat{F}_{\mathrm{can}} \circ \widehat{T}_{\mathrm{can}} \,.$$

Closing the circle: the d_{sdw} example

Proving soundness of the quantitative up-to context closure

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- The above theorems can be used to show that the distributive law ζ can be lifted to a distributive lifting between the Wasserstein liftings.
- Use the CSL-LICS'14 result to infer the soundness of the up-to technique.

Conclusions

Summary and future work

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- How does this relate to other “fibrational” approaches to functor liftings, e.g. the codensity liftings of Katsumata and Sato? Can we envisage a generic Kantorovich-Rubinstein duality?
- Future work: can we capture the work of Chatzikokolakis et. al. on up-to techniques for behavioural metrics in a probabilistic setting?