Up-to techniques for behavioural metrics
via fibrations

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Overview
In this talk

Part I

• Up-to techniques for behavioural equivalences
• Behavioural metrics coinductively
• A running example: computing distances between regular languages more efficiently

Part II

• A generic framework for proving soundness of up-to techniques using liftings of functors
• The Wasserstein lifting of a Set-functor
• Application to the running example
Part I: Introducing up-to techniques for behavioural metrics
Up-to techniques for behavioural equivalences

Used for proving behavioural equivalences of processes in concurrency theory:

- [Pous and Sangiorgi. 2011] *Enhancements of the coinductive proof method. In Advanced Topics in Bisimulation and Coinduction. Cambridge University Press*
Up-to techniques for behavioural equivalences

Used for proving behavioural equivalences of processes in concurrency theory:


Applications for automata: The HKC algorithm for checking language equivalence for NFAs

So what are up-to techniques?

In many cases behavioural equivalences are coinductive predicates, i.e., they can be expressed as the greatest fixpoint $\nu b$ of a monotone function

$$b: \text{Rel}_Q \rightarrow \text{Rel}_Q,$$

where $\text{Rel}_Q$ is the complete lattice of relations on the state space $Q$.

Coinduction proof principle:

$$\begin{align*}
(x, y) &\in r \quad r \subseteq b(r) \\
(x, y) &\in \nu b
\end{align*}$$
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**Coinduction up-to $f : \text{Rel}_Q \to \text{Rel}_Q$ proof principle:**

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where $\text{Rel}_Q$ is the complete lattice of relations on the state space $Q$.

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$$\frac{r \leq b(r)}{r \leq \nu b}$$

**Coinduction up-to $f : \text{Rel}_Q \to \text{Rel}_Q$ proof principle:**

$$\frac{r \leq b(f(r))}{r \leq \nu b}$$
Sound and Compatible up-to techniques

**Definition (Sound up-to technique)**
A function \( f : \text{Rel}_Q \to \text{Rel}_Q \) is sound w.r.t. \( \nu b \) when the coinduction up-to \( f \) proof principle is valid:

\[
\frac{(x, y) \in r \quad r \subseteq b(f(r))}{(x, y) \in \nu b}
\]

Soundness of up-to techniques is not a compositional notion and can be hard to establish.

**Definition (Compatible up-to technique)**
A monotone function \( f : \text{Rel}_Q \to \text{Rel}_Q \) is compatible w.r.t. \( \nu b \) when

\[
f \circ b(r) \subseteq b \circ f(r)
\]

for all relations \( r \).

Lemma: Compatibility implies soundness.
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Definition (Compatible up-to technique)
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Lemma: Compatibility implies soundness.
Example: language equivalence for NFAs

Example

Up-to congruence closure is a sound up-to technique w.r.t. language equivalence for determinized NFAs.
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Two crucial observations:

• Language equivalence coincides with bisimilarity for deterministic automata, i.e. is \( \nu b \) for \( b : \text{Rel}_Q \rightarrow \text{Rel}_Q \) given by

\[
b(r) = \{(x, y) \mid o(x) = o(y) \text{ and } \forall a \in A, (\delta_a(x), \delta_b(y)) \in r\}
\]
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\]

• The determinization of an NFA with states \( Q \) also has an algebraic structure: \( \mathcal{P}Q \) is a join-semilattice, and, moreover:

\[
L(X) \cup L(Y) = L(X \cup Y), \quad \text{for } X, Y \in \mathcal{P}Q.
\]
Example: language equivalence for NFAs

To prove that two states $x$ and $y$ in an NFA accept the same language it suffices to compute a bisimulation relating $\{x\}$ and $\{y\}$ in the determinized automaton.
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To prove that two states \( x \) and \( y \) in an NFA accept the same language it suffices to compute a bisimulation relating \( \{x\} \) and \( \{y\} \) in the determinized automaton.

Define **up-to congruence** as the map \( \text{cgr}: \text{Rel}_{PQ} \rightarrow \text{Rel}_{PQ} \) sending a relation \( r \) to its closure under equivalence and the rule:

\[
\begin{align*}
(X_1, Y_1) \in r & \quad (X_2, Y_2) \in r \\
\hline
(X_1 \cup X_2, Y_1 \cup Y_2) & \in r
\end{align*}
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(X_1, Y_1) \in r \quad (X_2, Y_2) \in r \\
(X_1 \cup X_2, Y_1 \cup Y_2) \in r
$$

The proof principle:

$$
(x, y) \in r \quad r \subseteq b(\text{cgr}(r)) \\
(x, y) \in \nu b
$$

is valid. The HKC algorithm [Bonchi and Pous, POPL’13] computes on-the-fly a bisimulation **up-to congruence** relating \{x\} and \{y\}.
Moving to behavioural metrics

In a quantitative setting, exact behavioural equivalences are not robust notions, remember Radu’s talk: “Probabilistic bisimulations are useless”.
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One replaces notions of equivalence between processes by notions of distances between processes, an idea originally due to Jou and Smolka 1990.
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We have seen how one can lift distances between states of a system to distances between probability distributions on these state spaces.
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We have seen how one can lift distances between states of a system to distances between probability distributions on these state spaces.

Computing behavioural metrics is not easy... Can we use up-to techniques? In the process we also discuss systematic liftings of arbitrary Set-functors to pseudo-metrics.
Running example: distance between regular languages

Definition (Shortest distinguishing word distance)  
Given two languages \( L \) and \( K \), define

\[
d_{sdw}(L, K) = c^{|w|},
\]

where \( c \) is a constant such that \( 0 < c < 1 \) and \( w \) is the shortest word which belongs to exactly one of the languages \( L, K \).
**Definition (Shortest distinguishing word distance)**
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where $c$ is a constant such that $0 < c < 1$ and $w$ is the shortest word which belongs to exactly one of the languages $L, K$.

**Example:** In the NFA below $d_{sdw}(x_0, y_0) \leq c^n$.

How can we prove such inequalities more efficiently?
Running example: distance between regular languages

For a deterministic automaton, the distance $d_{sdw}$ between the languages accepted by two states can be expressed as the greatest fixpoint $\nu b$ of a function

$$b: [0, 1]^{Q \times Q} \to [0, 1]^{Q \times Q}$$

defined on the complete lattice $[0, 1]^{Q \times Q}$ ordered with the reversed point-wise order $<:

$$b(d)(q_1, q_2) = \begin{cases} 
1, & \text{if only one of } q_1, q_2 \\
\cdot \max_{a \in A} d(\delta_a(q_1), \delta_a(q_2)), & \text{otherwise}
\end{cases}$$
Option 1 (coinduction): Determine the NFA and find a distance $\overline{d}$ such that $\overline{d}(\{x_0\}, \{y_0\}) \leq c^n$ and $\overline{d} < b(\overline{d})$. Use the coinduction principle:

$$\overline{d} < b(\overline{d}) \quad \Rightarrow \quad \overline{d} < \sqrt{b}$$
Option 1 (coinduction): Determine the NFA and find a distance $\bar{d}$ such that $\bar{d}(\{x_0\}, \{y_0\}) \leq c^n$ and $\bar{d} < b(\bar{d})$. Use the coinduction principle:

$$\bar{d} < b(\bar{d})$$

$$\bar{d} < \nu b$$

We obtain $\bar{d} < d_{sdw}$, and hence $d_{sdw}(\{x_0\}, \{y_0\}) \leq c^n$.  

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Running example: distance between regular languages

$$a, b \xrightarrow{a} x_0$$

$$a, b \xrightarrow{b} y_0$$

$$a, b \xrightarrow{a} x_1$$

$$a, b \xrightarrow{b} y_1$$

$$a, b \xrightarrow{a, b} x_2$$

$$a, b \xrightarrow{b} y_2$$

$$\cdots$$

$$a, b \xrightarrow{a} x_{n−1}$$

$$a, b \xrightarrow{b} y_{n−1}$$

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Option 1 (coinduction): Determine the NFA and find a distance $\bar{d}$ such that $\bar{d}(\{x_0\}, \{y_0\}) \leq c^n$ and $\bar{d} < b(\bar{d})$. Use the coinduction principle:

$\bar{d} < b(\bar{d})$

We obtain $\bar{d} < d_{sdw}$, and hence $d_{sdw}(\{x_0\}, \{y_0\}) \leq c^n$.

Disadvantage: we need to compute $\bar{d}$ for exponentially many pairs of states.
Option 2 (coinduction up-to): use a sound up-to context technique which closes a $[0,1]$-valued relation under the rules:

\[
\begin{align*}
d(X_1, X_2) & \leq r \\
\Rightarrow f(d)(X_1, X_2) & \leq r \\
f(d)(Y_1, Y_2) & \leq r
\end{align*}
\]

\[
\begin{align*}
f(d)(X_1, X_2) & \leq r \\
\Rightarrow f(d)(Y_1, Y_2) & \leq r \\
f(d)(X_1 \cup Y_1, X_2 \cup Y_2) & \leq r
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Option 2 (coinduction up-to): use a sound up-to context technique which closes a $[0,1]$-valued relation under the rules:

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\begin{align*}
\frac{d(X_1, X_2) \leq r}{f(d)(X_1, X_2) \leq r} & \quad \frac{f(d)(X_1, X_2) \leq r}{f(d)(Y_1, Y_2) \leq r} \quad \frac{f(d)(Y_1, Y_2) \leq r}{f(d)(X_1 \cup Y_1, X_2 \cup Y_2) \leq r}
\end{align*}
\]

Find a relaxed invariant $\overline{d}$ such that $\overline{d} < b(f(\overline{d}))$ and $\overline{d}([x_0], [y_0]) \leq c^n$. Use the coinduction up-to principle to conclude $\overline{d} < d_{sdw}$, and hence $d_{sdw}(x_0, y_0) \leq c^n$. 
Running example: distance between regular languages

Option 2 (coinduction up-to): use a sound up-to context technique which closes a \([0, 1]\)-valued relation under the rules:

\[
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\frac{d(X_1, X_2) \leq r}{f(d)(X_1, X_2) \leq r} & \quad \frac{f(d)(X_1, X_2) \leq r}{f(d)(Y_1, Y_2) \leq r} \\
\end{align*}
\]

Find a relaxed invariant \(\overline{d}\) such that \(\overline{d} < b(f(\overline{d}))\) and

\(\overline{d}(\{x_0\}, \{y_0\}) \leq c^n\). Use the coinduction up-to principle to conclude

\(\overline{d} < d_{sdw}\), and hence \(d_{sdw}(x_0, y_0) \leq c^n\).

Define \(\overline{d}(\{x_i\}, \{y_j\}) = c^{n-\max\{i,j\}}\) and \(\overline{d}(X, Y) = 1\) for all other values.

Notice that it suffices to define \(\overline{d}\) on a linear number of pairs.
Running example: distance between regular languages

From the generic framework developed in the rest of the talk, we will establish:

- how both $b$ and $f$ can be expressed in terms of so called Wasserstein liftings of functors to $[0,1]$-valued relations.
- why the soundness of $f$ follows from a generic framework developed previously for up-to techniques in a fibrational setting.
Part II: Soundness of up-to techniques for behavioural metrics
Use the fibrational framework of our previous CSL-LICS’2014 paper to prove the soundness of the quantitative version of the up-to congruence technique. To this end:

- Coinductive predicates (in particular, behavioural metrics) can be expressed via functor liftings
- Up-to techniques can also be expressed via functor liftings
- Discuss functor liftings and in particular what we will call the Wasserstein lifting of a functor
- Apply all this machinery in the example of $d_{sdw}$
We consider systems modelled as coalgebras for a functor $F : \text{Set} \to \text{Set}$, i.e. maps of the form $\xi : X \to FX$. 
Coinductive predicates via functor liftings

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Coinductive predicates describing properties of a coalgebra $\xi: X \to FX$ can be seen as post-fixpoints of a composite map $b$

$$b: \text{Rel}_X \xrightarrow{\overline{F}} \text{Rel}_{FX} \xrightarrow{\xi^{-1}} \text{Rel}_X$$
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where $\bar{F}$ is a “lifting” of $F$ mapping relations on $X$ to relations on $FX$ and for $R \subseteq FX \times FX$

$$(x, y) \in \xi^{-1}(R) \iff (\xi(x), \xi(y)) \in R$$
Example 1: language equivalence via functor liftings

Forgetting about the initial state, a DFA is a coalgebra for the functor $F X = 2 \times X^A$, i.e. a map of the form

$$\langle o, \delta \rangle: X \to 2 \times X^A,$$

with $o(q) = 1$ iff $q$ is accepting and $\delta(q)(a) = \delta_a(q)$. 
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Language equivalence is the largest fixpoint of the composite map

$$b : \text{Rel}_X \xrightarrow{\bar{F}} \text{Rel}_{FX} \xrightarrow{\langle o, \delta \rangle^{-1}} \text{Rel}_X$$
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Language equivalence is the largest fixpoint of the composite map

$$b : \text{Rel}_X \xrightarrow{\overline{F}} \text{Rel}_{FX} \xrightarrow{\langle o, \delta \rangle^{-1}} \text{Rel}_X$$

where $\overline{F}$ denotes here the so-called canonical lifting of $F$, i.e., for $R \subseteq X \times X$ and $(o_i, \phi_i) \in FX$ we have

$$(o_1, \phi_1) \overline{F}(R) (o_2, \phi_2)$$

iff

$$\begin{cases} o_1 = o_2 \\ \forall a \in A \quad \phi_1(a) R \phi_2(a) \end{cases}$$
Example 2: distance between regular languages

Forgetting about the initial state, a DFA is a coalgebra for the functor $FX = 2 \times X^A$, i.e. a map of the form

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The distance $d_{sdw}$ is the largest fixpoint of the monotone map $b$ on the lattice of $[0,1]$-valued relations, ordered by $\prec$ (the point-wise reverse order on the reals). This is obtained as the composite

$$b: [0,1]\text{-Rel}_X \xrightarrow{\bar{F}} [0,1]\text{-Rel}_{FX} \xrightarrow{\langle o, \delta \rangle^{-1}} [0,1]\text{-Rel}_X$$
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with $o(q) = 1$ iff $q$ is accepting and $\delta(q)(a) = \delta_a(q)$.

The distance $d_{sdw}$ is the largest fixpoint of the monotone map $b$ on the lattice of $[O, 1]$-valued relations, ordered by $<$ (the point-wise reverse order on the reals). This is obtained as the composite

$$b: [O, 1]\text{-Rel}_X \xrightarrow{\bar{F}} [O, 1]\text{-Rel}_{FX} \xrightarrow{\langle o, \delta \rangle^{-1}} [O, 1]\text{-Rel}_X$$

where $\bar{F}$ is defined for $d: X \times X \to [O, 1]$ by

$$(o_1, \phi_1) \bar{F}(d) (o_2, \phi_2) \text{ iff } \begin{cases} 1 & \text{if } o_1 \neq o_2 \\ c \cdot \max_{a \in A} d(\phi_1(a), \phi_2(a)) & \text{otherwise} \end{cases}$$
Up-to context closure via functor liftings

If we consider now a system, which is not only modelled as a coalgebra, but is also equipped with a compatible algebraic structure, it makes sense to consider the up-to congruence technique with respect to this algebraic structure.
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**Definition (Bialgebra)**
Consider two functors $F, T$ and a natural transformation $\zeta : TF \Rightarrow FT$. A bialgebra for $\zeta$ is a tuple $(X, \alpha, \xi)$ such that

$$\alpha : TX \rightarrow X$$

is a $T$-algebra,

$$\xi : X \rightarrow FX$$

is an $F$-coalgebra

so that the next diagram commutes.

$$\begin{array}{ccc} TX & \xrightarrow{\alpha} & X & \xrightarrow{\xi} & FX \\ \downarrow_{T\xi} & & \downarrow_{F\alpha} & & \uparrow_{\zeta X} \\ TFX & \xrightarrow{\zeta X} & FTX \end{array}$$
Bialgebras

Definition (Bialgebra)
Consider two functors $F, T$ and a natural transformation $\zeta : TF \Rightarrow FT$. A bialgebra for $\zeta$ is a tuple $(X, \alpha, \xi)$ such that $\alpha : TX \to X$ is a $T$-algebra, $\xi : X \to FX$ is an $F$-coalgebra so that the next diagram commutes.

\[
\begin{array}{ccc}
TX & \xrightarrow{\alpha} & X \\
\downarrow{T\xi} & & \uparrow{F\alpha} \\
TFX & \xrightarrow{\zeta_X} & FTX
\end{array}
\]

Example
The determinization of an NFA with states $Q$ is a bialgebra of the form $(\mathcal{P}Q, \cup, \gamma)$ for the functors $FX = 2 \times X^A$, $TX = \mathcal{P}X$ and $\zeta_X : \mathcal{P}(2 \times X^A) \to 2 \times (\mathcal{P}X)^A$ defined for $M \subseteq 2 \times X^A$ by

\[
\zeta_X(M) = (\bigvee_{(b,f) \in M} b, [a \mapsto \{f(a) \mid (b, f) \in M\}])
\]
Example 1: Up-to context closure for determinized NFAs

To sum up, a determinized NFA has both algebra and coalgebra structures, which are related by a distributive law:

\[ \cup : \mathcal{P}\mathcal{P} \mathcal{Q} \to \mathcal{P} \mathcal{Q} \quad \text{and} \quad \xi : \mathcal{P} \mathcal{Q} \to 2 \times (\mathcal{P} \mathcal{Q})^A. \]
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To sum up, a determinized NFA has both algebra and coalgebra structures, which are related by a distributive law:

\[ \cup : \mathcal{P} \mathcal{P} Q \to \mathcal{P} Q \quad \text{and} \quad \xi : \mathcal{P} Q \to 2 \times (\mathcal{P} Q)^A. \]

Context closure \( \text{ctx} : \text{Rel}_{\mathcal{P} Q} \to \text{Rel}_{\mathcal{P} Q} \) of a relation \( r \) on \( \mathcal{P} Q \) is defined via the rule:

\[
(X_1, Y_1) \in r \quad (X_2, Y_2) \in r \\
(\underbrace{X_1 \cup X_2}, \underbrace{Y_1 \cup Y_2}) \in \text{ctx}(r)
\]
Example 1: Up-to context closure for determinized NFAs

To sum up, a determinized NFA has both algebra and coalgebra structures, which are related by a distributive law:

\[ \cup: \mathcal{P}\mathcal{P}\mathcal{Q} \rightarrow \mathcal{P}\mathcal{Q} \quad \text{and} \quad \xi: \mathcal{P}\mathcal{Q} \rightarrow 2 \times (\mathcal{P}\mathcal{Q})^A. \]

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\[
\frac{(X_1, Y_1) \in r \quad (X_2, Y_2) \in r}{(X_1 \cup X_2, Y_1 \cup Y_2) \in \text{ctx}(r)}
\]

This can be seen as the composite map:

\[
\text{ctx}: \text{Rel}_{\mathcal{P}\mathcal{Q}} \xrightarrow{\bar{\mathcal{P}}} \text{Rel}_{\mathcal{P}\mathcal{P}\mathcal{Q}} \xrightarrow{\Sigma_\cup} \text{Rel}_{\mathcal{P}\mathcal{Q}}
\]

where \( \bar{\mathcal{P}} \) is the canonical relation lifting of \( \mathcal{P} \) and \( \Sigma_\cup \) is forward image along the \( \cup \).
Example 1: Up-to context closure for determinized NFAs

Context closure $\text{ctx}: \text{Rel}_{PQ} \rightarrow \text{Rel}_{PQ}$ of a relation $r$ on $PQ$ is defined via the rule:

$$(X_1, Y_1) \in r \quad (X_2, Y_2) \in r$$

$$(X_1 \cup X_2, Y_1 \cup Y_2) \in \text{ctx}(r)$$

This can be seen as the composite map:

$$\text{ctx}: \text{Rel}_{PQ} \xrightarrow{\overline{P}} \text{Rel}_{\overline{P}PQ} \xrightarrow{\Sigma_\cup} \text{Rel}_{PQ}$$

where $\overline{P}$ is the canonical relation lifting of $P$ and $\Sigma_\cup$ is forward image along the $\cup$, i.e., for $R \in \text{Rel}_{PQ}$ and $S \in \text{Rel}_{\overline{P}PQ}$:

- $(X, Y) \in \overline{P}(R)$ iff
  $$\forall A \in X, \exists B \in Y \quad (A, B) \in R$$
  $$\forall B \in Y, \exists A \in X \quad (A, B) \in R$$
- $(X, Y) \in \Sigma_\cup(S)$ iff $X = \cup X$, $Y = \cup Y$ and $(X, Y) \in S$. 

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Example 2: Quantitative up-to context closure for determinized NFAs

**Definition (Quantitative context closure)**
The quantitative context closure \( f: \text{Rel}_{PQ} \to \text{Rel}_{PQ} \) considered in the running example is defined as the composite

\[
f: [0, 1]-\text{Rel}_{PQ} \xrightarrow{\overline{P}} [0, 1]-\text{Rel}_{PPQ} \xrightarrow{\Sigma_{U}} [0, 1]-\text{Rel}_{PQ}
\]
Example 2: Quantitative up-to context closure for determinized NFAs

**Definition (Quantitative context closure)**
The quantitative context closure \( f: \text{Rel}_{PQ} \rightarrow \text{Rel}_{PQ} \) considered in the running example is defined as the composite

\[
\begin{align*}
f &: [0, 1]-\text{Rel}_{PQ} \xrightarrow{\overline{P}} [0, 1]-\text{Rel}_{P\cup P} \xrightarrow{\Sigma_{\cup}} [0, 1]-\text{Rel}_{PQ}
\end{align*}
\]

where \( \overline{P} \) is the “canonical” \([0, 1]\)-relation lifting of \( P \), equipping \( P \times X \) with the Hausdorff distance and \( \Sigma_{\cup} \) is forward image along the \( \cup \),
Definition (Quantitative context closure)
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$$f: [0, 1]-\text{Rel}_{PQ} \xrightarrow{\overline{P}} [0, 1]-\text{Rel}_{P\cup Q} \xrightarrow{\Sigma_{U}} [0, 1]-\text{Rel}_{PQ}$$

where $\overline{P}$ is the “canonical” $[0, 1]$-relation lifting of $P$, equipping $P \times X$ with the Hausdorff distance and $\Sigma_{U}$ is forward image along the $\cup$, i.e., for $d \in [0, 1]-\text{Rel}_{PQ}$ and $s \in [0, 1]-\text{Rel}_{P\cup Q}$:

- $\overline{P}(d)(X_1, X_2) = \sup \{ \sup \inf d(x_1, x_2), \sup \inf d(x_1, x_2) \}$
  
- $\Sigma_{U}(s)(X, Y) = \inf \{ s(X, Y) / \text{divides}.alt0 \cup X = X, \cup Y = Y \}$
Example 2: Quantitative up-to context closure for determinized NFAs

**Definition (Quantitative context closure)**
The quantitative context closure $f : \text{Rel}_{PQ} \to \text{Rel}_{PQ}$ considered in the running example is defined as the composite

$$f : [0, 1]-\text{Rel}_{PQ} \xrightarrow{\overline{P}} [0, 1]-\text{Rel}_{P\overline{P}Q} \xrightarrow{\Sigma_U} [0, 1]-\text{Rel}_{PQ}$$

where $\overline{P}$ is the “canonical” $[0, 1]$-relation lifting of $P$, equipping $P \times X$ with the Hausdorff distance and $\Sigma_U$ is forward image along the $\cup$, i.e., for $d \in [0, 1]-\text{Rel}_{PQ}$ and $s \in [0, 1]-\text{Rel}_{P\overline{P}Q}$:

- $\overline{P}(d)(X_1, X_2) = \sup \{ \sup_{x_1 \in X_1} \inf_{x_2 \in X_2} d(x_1, x_2), \sup_{x_2 \in X_2} \inf_{x_1 \in X_1} d(x_1, x_2) \}$
- $\Sigma_U(s)(X, Y) = \inf \{ s(\mathcal{X}, \mathcal{Y}) \mid \cup \mathcal{X} = X, \cup \mathcal{Y} = Y \}$.
A common framework for quantitative and qualitative setting

\[ \text{Rel}_X \xrightarrow{f} \text{Rel}_Y \]

Set

\[ \text{properties} \]

\[ \text{systems} \]
A common framework for quantitative and qualitative setting
A common framework for quantitative and qualitative setting

\[ [0,1]-\text{Rel}_X \quad [0,1]-\text{Rel}_Y \]

\[ X \xrightarrow{f} Y \]

\[ \Sigma \]

\[ \text{Set} \quad \text{systems} \]

\[ \text{properties} \quad p \]

\[ \text{ Set} \]

\[ \text{ systems} \]
A common framework for quantitative and qualitative setting

\[ [0, 1]-\text{Rel}_X \xrightarrow{\Sigma_f} [0, 1]-\text{Rel}_Y \]

Set properties

\[ [0, 1]-\text{Rel} \]

systems
A functor $p : \mathcal{P} \to \mathcal{B}$ is called a fibration when for every morphism $f : X \to Y$ in $\mathcal{B}$ and every $R$ in $\mathcal{P}$ with $p(R) = Y$ there exists a map $\tilde{f}_R : f^*(R) \to R$ such that $p(\tilde{f}_R) = f$ and satisfying the universal property:

For all maps $g : Z \to X$ in $\mathcal{B}$ and $u : Q \to R$ in $\mathcal{P}$ sitting above $fg$ (i.e., $p(u) = fg$) there is a unique map $v : Q \to f^*(R)$ such that $u = \tilde{f}_R v$ and $p(v) = g$. 
A functor $p : P \to B$ is called a fibration when for every morphism $f : X \to Y$ in $B$ and every $R$ in $P$ with $p(R) = Y$ there exists a map $\tilde{f}_R : f^*(R) \to R$ such that $p(\tilde{f}_R) = f$ and satisfying the universal property:

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A functor $p: \mathcal{P} \to \mathcal{B}$ is called a **fibration** when for every morphism $f: X \to Y$ in $\mathcal{B}$ and every $R$ in $\mathcal{P}$ with $p(R) = Y$ there exists a map $\tilde{f}_R: f^*(R) \to R$ such that $p(\tilde{f}_R) = f$ and satisfying the universal property:

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\[
\begin{align*}
\text{Q} & \xrightarrow{\forall u} & R \\
\text{f}^*(R) & \xrightarrow{\tilde{f}_R} & R \\
\text{Z} & \xrightarrow{fg} & \text{X} \\
\text{X} & \xrightarrow{f} & \text{Y}
\end{align*}
\]
A functor $p: \mathcal{P} \to \mathcal{B}$ is called a \textbf{fibration} when for every morphism $f: X \to Y$ in $\mathcal{B}$ and every $R$ in $\mathcal{P}$ with $p(R) = Y$ there exists a map $\tilde{f}_R: f^*(R) \to R$ such that $p(\tilde{f}_R) = f$ and satisfying the universal property:

\[
\begin{array}{ccc}
Q & \xrightarrow{\forall u} & f^*(R) \\
\downarrow & & \downarrow \tilde{f}_R \\
& X & \xrightarrow{f} Y
\end{array}
\]

For all maps $g: Z \to X$ in $\mathcal{B}$ and $u: Q \to R$ in $\mathcal{P}$ sitting above $fg$ (i.e., $p(u) = fg$) there is a unique map $v: Q \to f^*(R)$ such that $u = \tilde{f}_Rv$ and $p(v) = g$. 

In a bi/fibration every reindexing has a left adjoint $\Sigma_f/\text{univ}_f$. 
A functor \( p: \mathcal{P} \to \mathcal{B} \) is called a \textbf{fibration} when for every morphism \( f: X \to Y \) in \( \mathcal{B} \) and every \( R \) in \( \mathcal{P} \) with \( p(R) = Y \) there exists a map \( \tilde{f}_R: f^*(R) \to R \) such that \( p(\tilde{f}_R) = f \) and satisfying the universal property:

For all maps \( g: Z \to X \) in \( \mathcal{B} \) and \( u: Q \to R \) in \( \mathcal{P} \) sitting above \( fg \) (i.e., \( p(u) = fg \)) there is a unique map \( v: Q \to f^*(R) \) such that \( u = \tilde{f}_R v \) and \( p(v) = g \).

In a \textbf{bifibration} every reindexing has a left adjoint \( \Sigma_f \vdash f^* \).
Liftings of functors

Given a fibration \( p: \mathbb{P} \rightarrow \mathbb{B} \) and a functor \( F: \mathbb{B} \rightarrow \mathbb{B} \), a lifting of \( F \) is a functor \( \hat{F}: \mathbb{P} \rightarrow \mathbb{P} \) such that

\[
P \xrightarrow{\hat{F}} \mathbb{P} \quad \xleftarrow{p} \quad \mathbb{B} \xrightarrow{F} \mathbb{B}
\]

For every \( r \in \mathbb{P} \) and \( f: \mathbb{X} \rightarrow \mathbb{Y} \) in \( \mathbb{B} \), we have a canonical natural transformation

\[
F \circ f^* (R) \rightarrow (Ff \circ F)(R)
\]

The lifting \( \hat{F} \) is called a fibred lifting of \( F \) when the above natural transformation is an isomorphism for every \( r \).
Liftings of functors

Given a fibration \( p: \mathcal{P} \to \mathcal{B} \) and a functor \( F: \mathcal{B} \to \mathcal{B} \), a lifting of \( F \) is a functor \( \widehat{F}: \mathcal{P} \to \mathcal{P} \) such that

\[
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{\widehat{F}} & \mathcal{P} \\
p & \downarrow & \downarrow p \\
\mathcal{B} & \xrightarrow{F} & \mathcal{B}
\end{array}
\]

For every \( r \in \mathcal{P}_Y \) and \( f: \mathcal{X} \to \mathcal{Y} \) in \( \mathcal{B} \), we have a canonical natural transformation

\[
\overline{F} \circ f^*(R) \to (Ff)^* \circ \overline{F}(R).
\]

The lifting \( \widehat{F} \) is called a fibred lifting of \( F \) when the above natural transformation is an isomorphism for every \( r \).
Consider a fibration $p: \mathbb{P} \to \mathbb{B}$ and a bialgebra in $\mathbb{B}$

\[ TX \xrightarrow{\alpha} X \xrightarrow{\gamma} FX \]

\[ T \gamma \downarrow \quad \quad \quad \quad \quad F \alpha \uparrow \]

\[ TFX \xrightarrow{\zeta_X} FTX \]

Consider two liftings $\overline{T}$ and $\overline{F}$ of $T$, respectively $F$, to the category $\mathbb{P}$ such that there exists a lifting $\overline{\zeta}: \overline{T} \circ \overline{F} \Rightarrow \overline{F} \circ \overline{T}$ of the distributive law $\zeta$. Then the up-to technique

\[ f: \mathbb{P}_X \xrightarrow{\overline{T}} \mathbb{P}_{TX} \xrightarrow{\Sigma_\alpha} \mathbb{P}_X \]

is sound with respect to the coinductive predicate defined by

\[ b: \mathbb{P}_X \xrightarrow{\overline{F}} \mathbb{P}_{FX} \xrightarrow{\xi^*} \mathbb{P}_X \]
Lifting of functors to many-valued relations
Quantale valued predicates and relations

**Definition**
A quantale $\mathcal{V}$ is a complete lattice equipped with an associative operation $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ which is distributive on both sides over arbitrary joins $\lor$. We assume the tensor is commutative and has a unit $1$. 
Quantale valued predicates and relations

Definition
A quantale $\mathcal{V}$ is a complete lattice equipped with an associative operation $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ which is distributive on both sides over arbitrary joins $\bigvee$. We assume the tensor is commutative and has a unit $1$.

Definition
Given a set $X$ and a quantale $\mathcal{V}$, a $\mathcal{V}$-valued predicate on $X$ is a map $p : X \to \mathcal{V}$. A $\mathcal{V}$-valued relation on $X$ is a map $r : X \times X \to \mathcal{V}$. 
Quantale valued predicates and relations

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Given two $\mathcal{V}$-valued predicates $p, q : X \to \mathcal{V}$, we say that

$$p \leq q \iff \forall x \in X. \ p(x) \leq q(x).$$
Quantale valued predicates and relations

Definition
A quantale $\mathcal{V}$ is a complete lattice equipped with an associative operation $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ which is distributive on both sides over arbitrary joins $\vee$. We assume the tensor is commutative and has a unit $1$.

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Given a set $X$ and a quantale $\mathcal{V}$, a $\mathcal{V}$-valued predicate on $X$ is a map $p : X \to \mathcal{V}$. A $\mathcal{V}$-valued relation on $X$ is a map $r : X \times X \to \mathcal{V}$.

Given two $\mathcal{V}$-valued predicates $p, q : X \to \mathcal{V}$, we say that

$$p \leq q \iff \forall x \in X. p(x) \leq q(x).$$

A morphism between $\mathcal{V}$-valued predicates $p : X \to \mathcal{V}$ and $q : Y \to \mathcal{V}$ is a map $f : X \to Y$ such that $p \leq q \circ f$. We consider the category $\mathcal{V}$-Pred whose objects are $\mathcal{V}$-valued predicates and arrows are as above.
The bifibrations of \( \mathcal{V} \)-valued predicates

\[
\mathcal{V}\text{-Pred}_X \xrightarrow{f} \mathcal{V}\text{-Pred}_Y
\]

\[
\mathcal{V}\text{-Pred} \xrightarrow{p} \text{Set}
\]

properties systems
The bifibrations of $\mathcal{V}$-valued predicates

\[ f^* (p) = p \circ f, \]
The bifibrations of $\forall$-valued predicates

$\forall\text{-Pred}_X \xrightarrow{\Sigma_f} \forall\text{-Pred}_Y$

$f^*$

$X \xrightarrow{f} Y$

$\forall\text{-Pred} \xrightarrow{p} \text{Set}$

properties

systems

- $f^*(p) = p \circ f$,
- $\Sigma_f(p)(y) = \bigvee \{p(x) \mid x \in f^{-1}(y)\}$
The bifibrations of $\mathcal{V}$-valued predicates and relations

We have a change-of-base situation, where $\Delta X = X \times X$.

We obtain a bifibration $\mathcal{V}$-Rel $\rightarrow$ Set.

Remark: $\mathcal{V}$-categories are $\mathcal{V}$-valued relations $r : X \times X \rightarrow \mathcal{V}$ that are
• reflexive if for all $x \in X$ we have $r(x, x) \geq \text{false}$, and
• transitive if for all $x, y, z \in X$ we have $r(y, z) \otimes r(x, y) \leq r(x, y)$.

We also obtain a bifibration $\mathcal{V}$-Cat $\rightarrow$ Set.

For the quantale $([\text{false}, \infty], \geq)$ these are the generalized pseudometrics from Lawvere's seminal paper.
The bifibrations of \( \mathcal{V} \)-valued predicates and relations

We have a change-of-base situation, where \( \Delta X = X \times X \).

\[
\begin{aligned}
\mathcal{V}\text{-Rel} & \xrightarrow{l} \mathcal{V}\text{-Pred} \\
\downarrow & \quad \downarrow \\
\text{Set} & \xrightarrow{\Delta} \text{Set}
\end{aligned}
\]

We obtain a bifibration \( \mathcal{V}\text{-Rel} \to \text{Set} \).

**Remark:** \( \mathcal{V} \)-categories are \( \mathcal{V} \)-valued relation \( r : X \times X \to \mathcal{V} \) that are

- reflexive if for all \( x \in X \) we have \( r(x, x) \geq 1 \), and
- transitive if for all \( x, y, z \in X \) we have \( r(y, z) \otimes r(x, y) \leq r(x, y) \).

We also obtain a bifibration \( \mathcal{V}\text{-Cat} \to \text{Set} \).
The bifibrations of $\mathcal{V}$-valued predicates and relations

We have a change-of-base situation, where $\Delta X = X \times X$.

$$
\begin{array}{ccc}
\mathcal{V}\text{-Rel} & \overset{\iota}{\longrightarrow} & \mathcal{V}\text{-Pred} \\
\downarrow & & \downarrow \\
\text{Set} & \overset{\Delta}{\longrightarrow} & \text{Set}
\end{array}
$$

We obtain a bifibration $\mathcal{V}\text{-Rel} \to \text{Set}$.

**Remark:** $\mathcal{V}$-categories are $\mathcal{V}$-valued relation $r : X \times X \to \mathcal{V}$ that are

- **reflexive** if for all $x \in X$ we have $r(x, x) \geq 1$, and
- **transitive** if for all $x, y, z \in X$ we have $r(y, z) \otimes r(x, y) \leq r(x, y)$.

We also obtain a bifibration $\mathcal{V}\text{-Cat} \to \text{Set}$.

For the quantale $([0, \infty], \geq_{[0, \infty]}, +, 0)$ these are the generalized pseudo-metrics from Lawvere's 1973 seminal paper.
A systematic way of lifting functors

Step 1: Lift a Set-functor $F$ to a functor $\hat{F}$ in the category of $\forall$-predicates.
A systematic way of lifting functors

\[ \mathcal{V}\text{-Rel} \xrightarrow{l} \mathcal{V}\text{-Pred} \]

\[ \downarrow \quad \downarrow \]

\[ \text{Set} \xrightarrow{\Delta} \text{Set} \]

**Step 1:** Lift a Set-functor \( F \) to a functor \( \tilde{F} \) the category of \( \mathcal{V} \)-predicates.

**Step 2:** Transfer this lifting to a lifting \( \bar{F} \) on \( \mathcal{V} \)-relations.
A systematic way of lifting functors

\[
\begin{array}{c}
\nu\text{-Rel} \xrightarrow{\iota} \nu\text{-Pred} \\
\downarrow \quad \downarrow \\
\text{Set} \xrightarrow{\Delta} \text{Set}
\end{array}
\]

**Step 1:** Lift a Set-functor \( F \) to a functor \( \widehat{F} \) the category of \( \nu \)-predicates.

**Step 2:** Transfer this lifting to a lifting \( \overline{F} \) on \( \nu \)-relations.

**Step 3:** When does \( \overline{F} \) restrict to \( \nu \)-categories?
A systematic way of lifting functors

\[ \mathcal{V}-\text{Rel} \xrightarrow{l} \mathcal{V}-\text{Pred} \]
\[ \downarrow \quad \downarrow \]
\[ \text{Set} \xrightarrow{\Delta} \text{Set} \]

**Step 1:** Lift a Set-functor \( F \) to a functor \( \hat{F} \) on the category \( \mathcal{V}-\text{Pred} \).

**Proposition.** There is a one-to-one correspondence between

- fibred liftings \( \overline{F} \) of \( F \) to \( \mathcal{V}-\text{Pred} \),
- monotone natural transformations \( \mathcal{V}^{-} \Rightarrow \mathcal{V}^{F^{-}} \),
- monotone evaluation maps \( \text{ev} : F\mathcal{V} \rightarrow \mathcal{V} \).

We also define \( \text{ev}_{\text{can}} : F\mathcal{V} \rightarrow \mathcal{V} \) as follows:

\[ \text{ev}_{\text{can}}(u) = \bigvee \{ r \mid u \in F(\uparrow r) \}. \]
A systematic way of lifting functors

\[ \mathcal{V} \text{-Rel} \xrightarrow{\lambda} \mathcal{V} \text{-Pred} \]

\[ \text{Set} \xrightarrow{\Delta} \text{Set} \]

**Step 2:** Transfer the predicate lifting \( \hat{F} \) to a lifting \( \overline{F} \) on \( \mathcal{V} \)-relations.

\[ \mathcal{V} \text{-Rel}_X \xrightarrow{\lambda_X} \mathcal{V} \text{-Rel}_F \]

\[ \mathcal{V} \text{-Pred}_{\Delta X} \xrightarrow{\overline{F}_{\Delta X}} \mathcal{V} \text{-Pred}_{F \Delta X} \xrightarrow{\Sigma \lambda_X} \mathcal{V} \text{-Pred}_{\Delta FX} \]

where \( \lambda_X : F(X \times X) \rightarrow FX \times FX \) is given by the pairing of projections \( \langle F\pi_1, F\pi_2 \rangle \).
Step 2: Transfer the predicate lifting $\widehat{F}$ to a lifting $\overline{F}$ on $\mathcal{V}$-relations.

Concretely, the lifting $\overline{F}$ is defined via “couplings”:

$$\overline{F}(p)(t_1, t_2) = \bigvee \{\widehat{F}(p)(t) \mid t \in F(X \times X), F\pi_i(t) = t_i\}$$

We call $\overline{F}$ the Wasserstein lifting associated with $\widehat{F}$. 
A systematic way of lifting functors

Step 3: When does $\widetilde{F}$ restrict to $\mathcal{V}$-categories?

We have the following characterization theorem, where $\kappa_X$ denotes the constant to 1 predicate on $X$, and for two predicates $p, q : X \to \mathcal{V}$ we denote by $p \otimes q : X \to \mathcal{V}$ the predicate mapping $x$ to $p(x) \otimes q(x)$.

Theorem. Assume $\widetilde{F}$ is a lifting of $F$ to $\mathcal{V}$-Pred and $\widetilde{F}$ is the corresponding $\mathcal{V}$-Rel Wasserstein lifting. Then

- If $\widetilde{F}(\kappa_X) \geq \kappa_{FX}$ then $\widetilde{F}$ preserves reflexive relations,
**A systematic way of lifting functors**

**Step 3:** When does $\bar{F}$ restrict to $\mathcal{V}$-categories?

We have the following characterization theorem, where $\kappa_X$ denotes the constant to 1 predicate on $X$, and for two predicates $p, q : X \to \mathcal{V}$ we denote by $p \otimes q : X \to \mathcal{V}$ the predicate mapping $x$ to $p(x) \otimes q(x)$.

**Theorem.** Assume $\widehat{F}$ is a lifting of $F$ to $\mathcal{V}$-Pred and $\bar{F}$ is the corresponding $\mathcal{V}$-Rel Wasserstein lifting. Then

- If $\widehat{F}(\kappa_X) \geq \kappa_{FX}$ then $\bar{F}$ preserves reflexive relations,
- If $\widehat{F}$ is a fibred lifting, $F$ preserves weak pullbacks and $\widehat{F}(p \otimes q) \geq \widehat{F}(p) \otimes \widehat{F}(q)$ then $\bar{F}$ preserves transitive relations,
A systematic way of lifting functors

Step 3: When does $\overline{F}$ restrict to $\mathcal{V}$-categories?

We have the following characterization theorem, where $\kappa_X$ denotes the constant to 1 predicate on $X$, and for two predicates $p, q: X \to \mathcal{V}$ we denote by $p \otimes q: X \to \mathcal{V}$ the predicate mapping $x$ to $p(x) \otimes q(x)$.

**Theorem.** Assume $\widehat{F}$ is a lifting of $F$ to $\mathcal{V}$-Pred and $\overline{F}$ is the corresponding $\mathcal{V}$-Rel Wasserstein lifting. Then

- If $\widehat{F}(\kappa_X) \geq \kappa_{FX}$ then $\overline{F}$ preserves reflexive relations,
- If $\widehat{F}$ is a fibred lifting, $F$ preserves weak pullbacks and $\widehat{F}(p \otimes q) \geq \widehat{F}(p) \otimes \widehat{F}(q)$ then $\overline{F}$ preserves transitive relations,
- $\overline{F}$ preserves symmetric relations.
A systematic way of lifting functors

**Step 3:** When does $\bar{F}$ restrict to $\mathcal{V}$-categories?

We have the following characterization theorem, where $\kappa_X$ denotes the constant to 1 predicate on $X$, and for two predicates $p, q: X \to \mathcal{V}$ we denote by $p \otimes q: X \to \mathcal{V}$ the predicate mapping $x$ to $p(x) \otimes q(x)$.

**Theorem.** Assume $\widehat{F}$ is a lifting of $F$ to $\mathcal{V}$-Pred and $\bar{F}$ is the corresponding $\mathcal{V}$-Rel *Wasserstein lifting*. Then

- If $\widehat{F}(\kappa_X) \geq \kappa_{FX}$ then $\bar{F}$ preserves reflexive relations,
- If $\widehat{F}$ is a fibred lifting, $F$ preserves weak pullbacks and $\widehat{F}(p \otimes q) \geq \widehat{F}(p) \otimes \widehat{F}(q)$ then $\bar{F}$ preserves transitive relations,
- $\bar{F}$ preserves symmetric relations.

Whenever $F$ preserves weak pullbacks the canonical evaluation lifting $\widehat{F}_{can}$ satisfies the above conditions.
Theorem
Assume the natural transformation $\zeta: T \circ F \Rightarrow F \circ T$ lifts to a natural transformation $\hat{\zeta}: \hat{T} \circ \hat{F} \Rightarrow \hat{F} \circ \hat{T}$ between $V$-predicate liftings and that we have $\hat{T} \circ \sum \chi_x^F \leq \sum T \chi_x^F \circ \hat{T}$. Then $\zeta$ lifts to a distributive law $\bar{\zeta}: \bar{T} \circ \bar{F} \Rightarrow \bar{F} \circ \bar{T}$ between the corresponding Wasserstein liftings.
Lifting distributive laws to Wasserstein liftings

**Theorem**
Assume the natural transformation \( \zeta : T \circ F \Rightarrow F \circ T \) lifts to a natural transformation \( \hat{\zeta} : \hat{T} \circ \hat{F} \Rightarrow \hat{F} \circ \hat{T} \) between \( \mathcal{V} \)-predicate liftings and that we have \( \hat{T} \circ \Sigma_{X}^{F} \leq \Sigma_{T}^{X} \circ \hat{T} \). Then \( \zeta \) lifts to a distributive law \( \overline{\zeta} : \overline{T} \circ \overline{F} \Rightarrow \overline{F} \circ \overline{T} \) between the corresponding Wasserstein liftings.

**Theorem**
Assume that \( \zeta : T \circ F \Rightarrow F \circ T \) is a natural transformation and that, furthermore, \( T \) preserves weak pullbacks and \( F \) preserves intersections. Then \( \zeta \) lifts to a natural transformation

\[
\hat{\zeta} : \hat{T}_{\text{can}} \circ \hat{F}_{\text{can}} \Rightarrow \hat{F}_{\text{can}} \circ \hat{T}_{\text{can}} .
\]
Closing the circle: the $d_{sdw}$ example
The determinization of an NFA is a bialgebra for a distributive law \( \zeta : \mathcal{P}(2 \times X^A) \rightarrow 2 \times (\mathcal{P}X)^A \).
• The determinization of an NFA is a bialgebra for a distributive law \( \zeta : \mathcal{P}(2 \times X^A) \rightarrow 2 \times (\mathcal{P}X)^A \).

• The coinductive predicates \( b \) and the up-to technique \( f \) can be described using suitable Wasserstein liftings.
• The determinization of an NFA is a bialgebra for a distributive law $\zeta: \mathcal{P}(2 \times X^A) \to 2 \times (\mathcal{P}X)^A$.

• The coinductive predicates $b$ and the up-to technique $f$ can be described using suitable Wasserstein liftings.

• The above theorems can be used to show that the distributive law $\zeta$ can be lifted to a distributive lifting between the Wasserstein liftings.
Proving soundness of the quantitative up-to context closure

• The determinization of an NFA is a bialgebra for a distributive law $\zeta: \mathcal{P}(2 \times X^A) \to 2 \times (\mathcal{P}X)^A$.

• The coinductive predicates $b$ and the up-to technique $f$ can be described using suitable Wasserstein liftings.

• The above theorems can be used to show that the distributive law $\zeta$ can be lifted to a distributive lifting between the Wasserstein liftings.

• Use the CSL-LICS’14 result to infer the soundness of the up-to technique.
Conclusions
Summary and future work

- We proved soundness of a quantitative version of the up-to context closure technique.

- We introduced a systematic definition and analysis of the Wasserstein lifting using fibrations, generalizing previous work on pseudo-metrics by Baldan et al.

- How does this relate to other fibrational approaches to functor liftings, e.g. the codensity liftings of Katsumata and Sato? Can we envisage a generic Kantorovich-Rubinstein duality?

- Future work: can we capture the work of Chatzikokolakis et al. on up-to techniques for behavioural metrics in a probabilistic setting?
Summary and future work

• We proved soundness of a quantitative version of the up-to context closure technique.
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• We proved soundness of a quantitative version of the up-to context closure technique.
• We introduced a systematic definition and analysis of the Wasserstein lifting using fibrations, generalizing previous work on pseudo-metrics by Baldan et. al.
• How does this relate to other “fibrational” approaches to functor liftings, e.g. the codensity liftings of Katsumata and Sato? Can we envisage a generic Kantorovich-Rubinstein duality?
• Future work: can we capture the work of Chatzikokolakis et. al. on up-to techniques for behavioural metrics in a probabilistic setting?
Summary and future work

• We proved soundness of a quantitative version of the up-to-context closure technique.
• We introduced a systematic definition and analysis of the Wasserstein lifting using fibrations, generalizing previous work on pseudo-metrics by Baldan et. al.
• How does this relate to other “fibrational” approaches to functor liftings, e.g. the codensity liftings of Katsumata and Sato? Can we envisage a generic Kantorovich-Rubinstein duality?
• Future work: can we capture the work of Chatzikokolakis et. al. on up-to techniques for behavioural metrics in a probabilistic setting?