Geometric semantics for asynchronous computability

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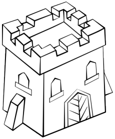
École Polytechnique

CHoCoLa
June 6, 2019

A topological approach for asynchronous computability

The two generals problem

Two divisions of the same army, commanded by general A and general B, are surrounding an enemy fortress.

A  B

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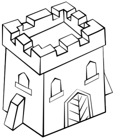

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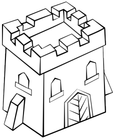

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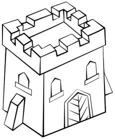

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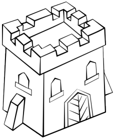

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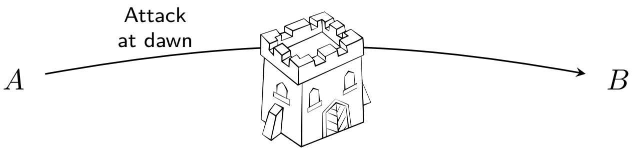

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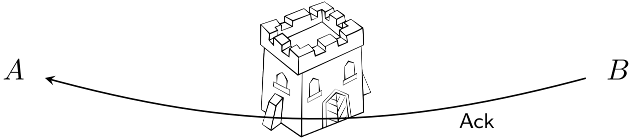

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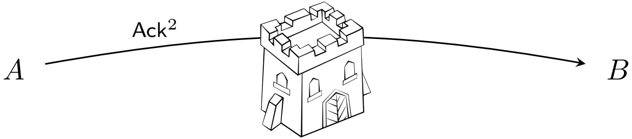

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a.k.a. Fault-tolerant distributed computing

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Remark: Usually, impossibility results come from a lack of information about the system, not from a lack of computing power.

A topological approach

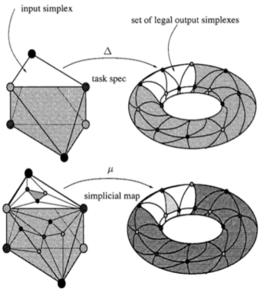

FIG. 13. Asynchronous computability theorem.

THEOREM 3.1 (ASYNCHRONOUS COMPUTABILITY THEOREM). *A decision task $\langle \mathcal{I}, \mathcal{O}, \Delta \rangle$ has a wait-free protocol using read-write memory if and only if there exists a chromatic subdivision σ of $\mathcal{I}$ and a color-preserving simplicial map*

$$\mu: \sigma(\mathcal{I}) \to \mathcal{O}$$

such that for each simplex S in $\sigma(\mathcal{I})$, $\mu(S) \in \Delta(carrier(S, \mathcal{I}))$.

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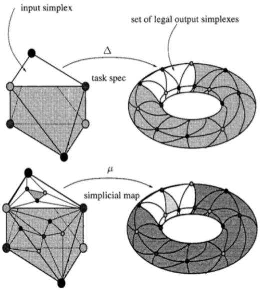

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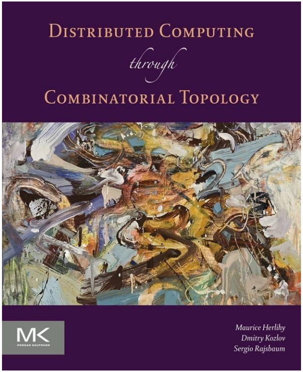

Herlihy, Kozlov, Rajsbaum, 2013

Simplicial complexes

Definition

An (abstract) **simplicial complex** is a pair $\langle V, S\rangle$ where V is a set of *vertices* and S is a downward-closed family of subsets of V called *simplices* (i.e., $X \in S$ and $Y \subseteq X$ implies $Y \in S$).

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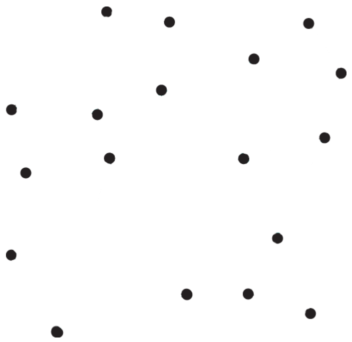

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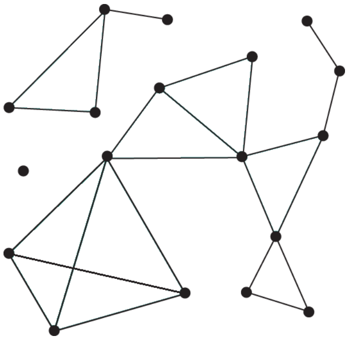

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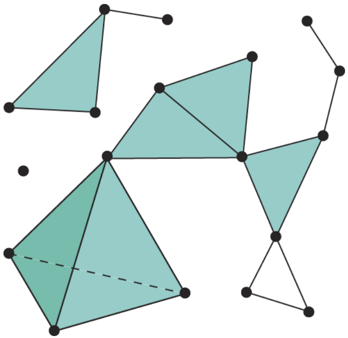

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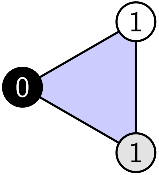

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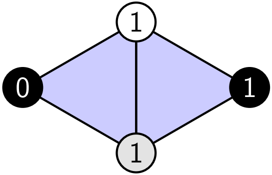

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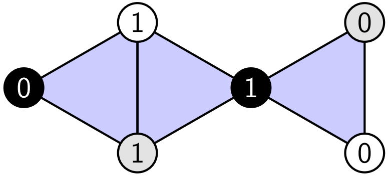

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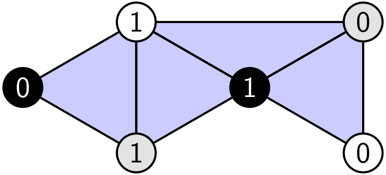

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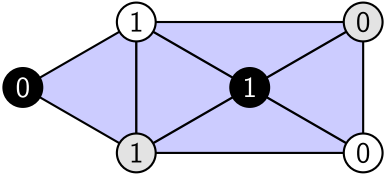

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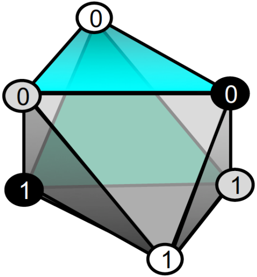

The immediate snapshot object

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immediate_snapshot :  'a -> 'a array
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Fix a number n of processes.
We suppose given a shared array A of size n.
Only process P_i can write in $A[i]$, but everyone can read it.

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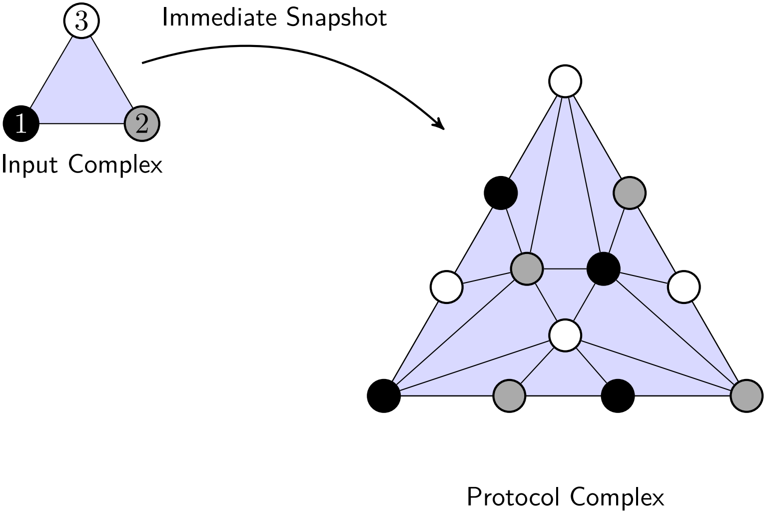

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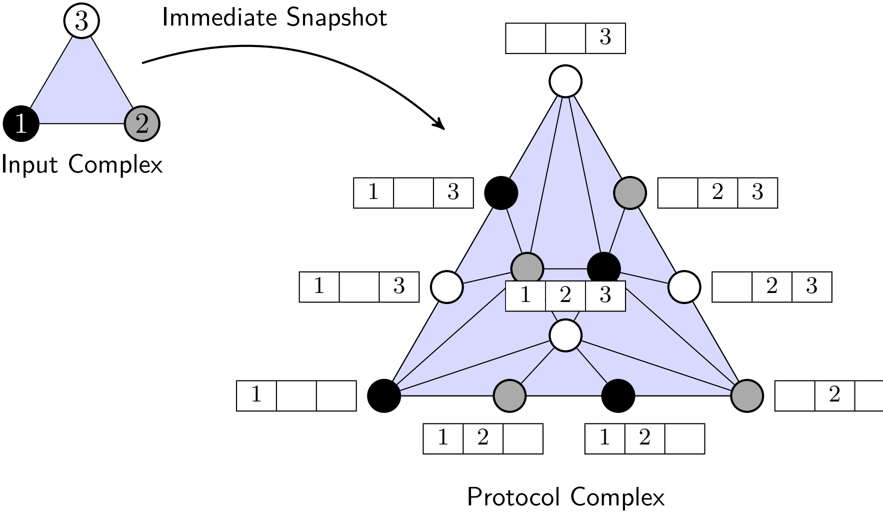

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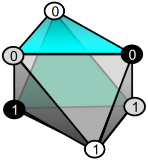
Examples: for 3 processes

- if the inputs are $(0, 0, 0)$, the outputs must be $(0, 0, 0)$.
- if the inputs are $(1, 0, 1)$, the outputs can be either $(0, 0, 0)$ or $(1, 1, 1)$.

Topological definition of task solvability



Input complex

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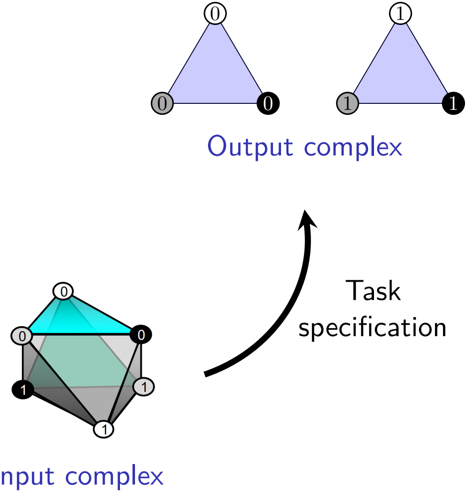

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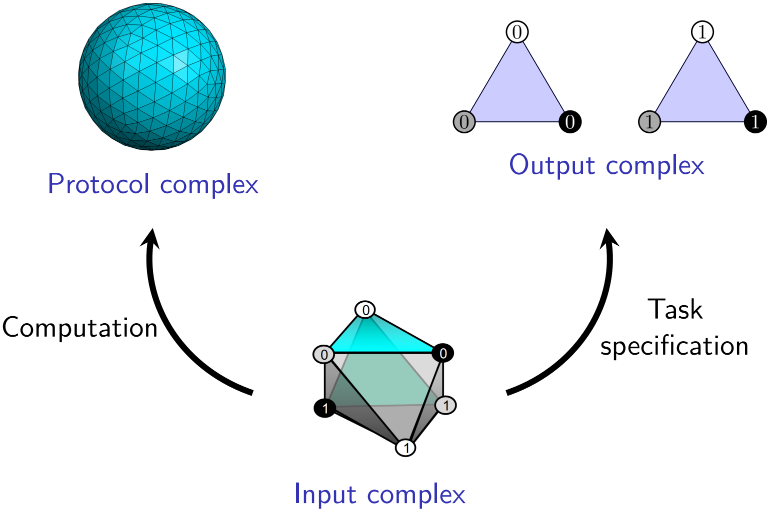

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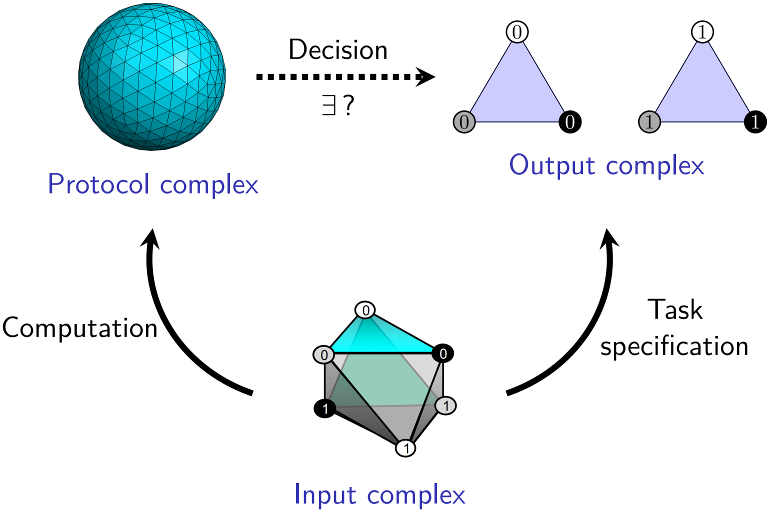

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Theorem (Herlihy and Shavit, 1999)

A task is solvable by a **wait-free** protocol using **read/write** registers if and only if there is a decision map from **a subdivision of** the input complex into the output complex such that [...].

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Goal: an asynchronous computability theorem for any objects.

Specifying concurrent objects

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$\longrightarrow$ We would like a composable notion of "solving".

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$\longrightarrow$ How do we specify a concurrent object?
$\longrightarrow$ What does it mean to implement an object?

Concurrent specifications

Idea: the specification of an object is the set of all the correct execution traces (Lamport, 1986).

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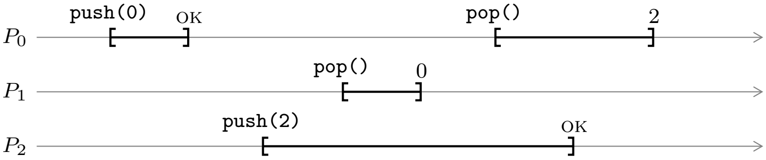

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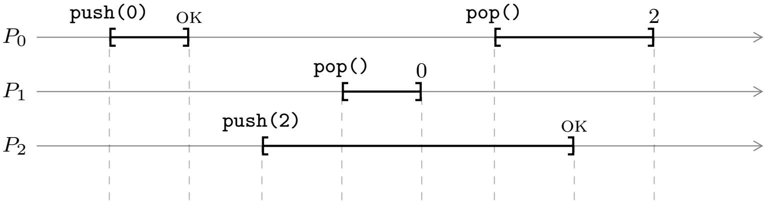
Idea: the specification of an object is the set of all the correct execution traces (Lamport, 1986).



$$T = \mathsf{i}_0^{\mathtt{push},0} \cdot \mathsf{r}_0^{\mathrm{OK}} \cdot \mathsf{i}_2^{\mathtt{push},2} \cdot \mathsf{i}_1^{\mathtt{pop}} \cdot \mathsf{r}_1^{2} \cdot \mathsf{i}_0^{\mathtt{pop}} \cdot \mathsf{r}_2^{\mathrm{OK}} \cdot \mathsf{r}_0^{0}$$

Trace formalism:

- Time is abstracted away.
- Alternation of invocations and responses on each process.

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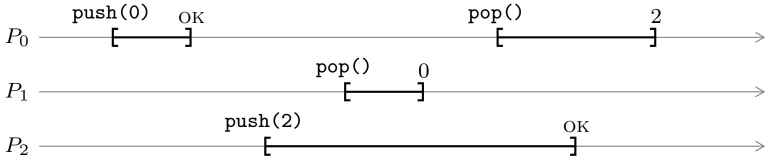

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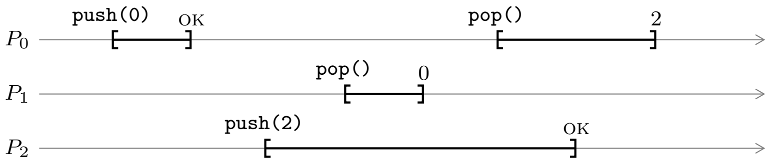

Write $\mathcal{T}$ for the set of all execution traces.

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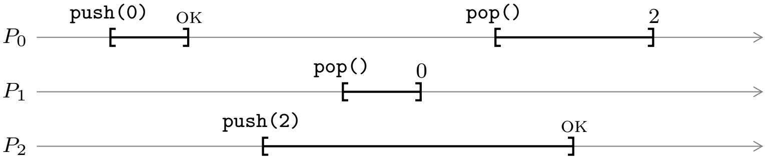

Write $\mathcal{T}$ for the set of all execution traces.

- A *concurrent specification* is a subset $\sigma \subseteq \mathcal{T}$.
- A program implements a specification σ if all the traces that it can produce belong to σ.

Linearizability (Herlihy & Wing, 1990)

$$\text{SeqSpec} \xrightarrow{\text{Lin}} \text{ConcSpec}$$

- **Input:** a sequential specification σ (e.g. list, queue, ...).
- **Output:** a concurrent specification $\text{Lin}(\sigma)$.

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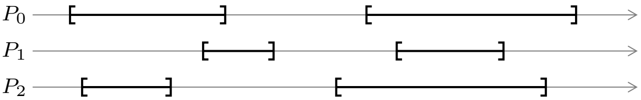

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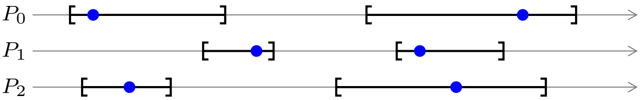

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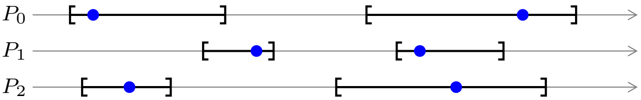

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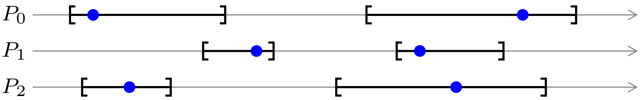

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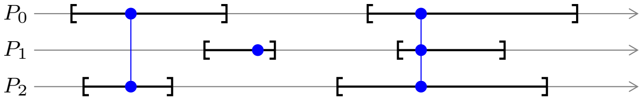
Some objects are not linearizable!

Their specification cannot be expressed as $\text{Lin}(\sigma)$, for any σ.

Concurrent variants of linearizability

Set-linearizability (Neiger, 1994)



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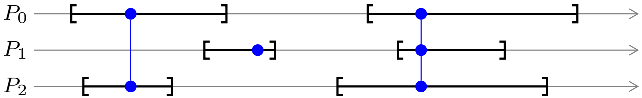

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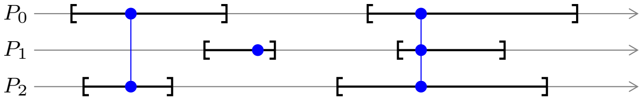

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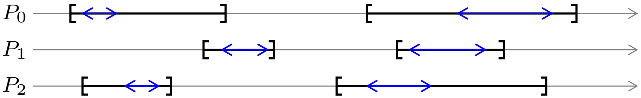

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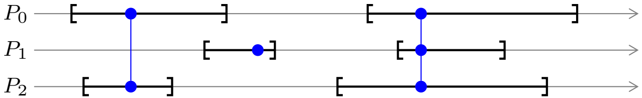

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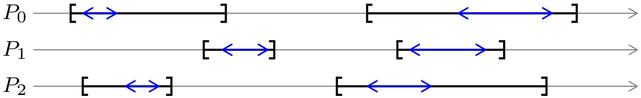

- Can specify every task!

Overview

Concurrent specifications

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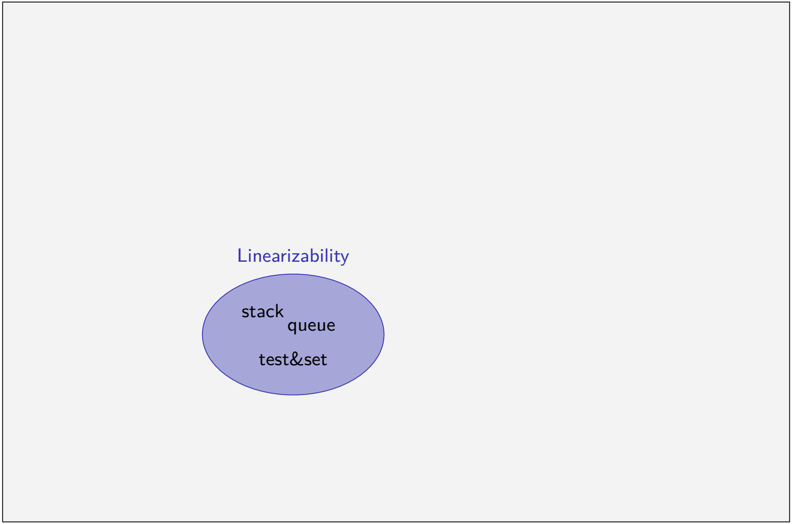

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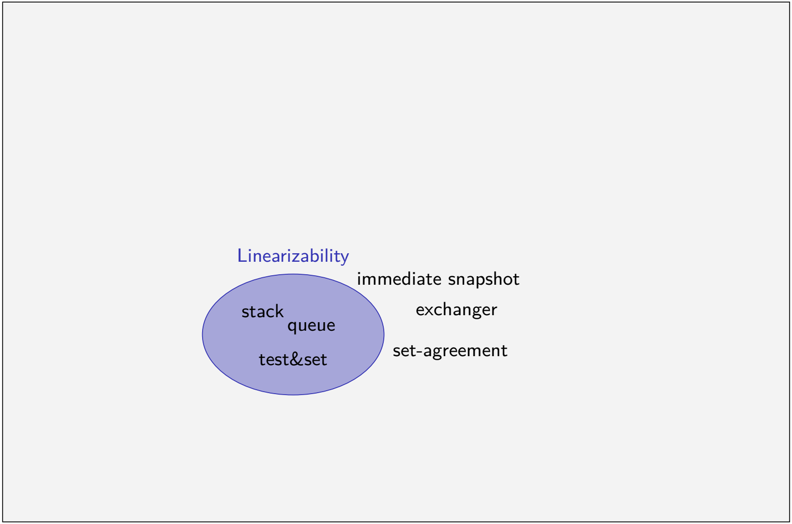

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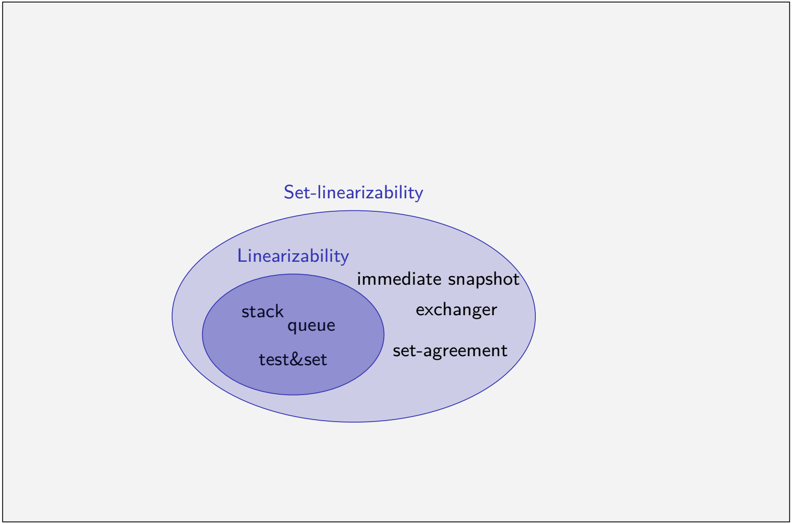

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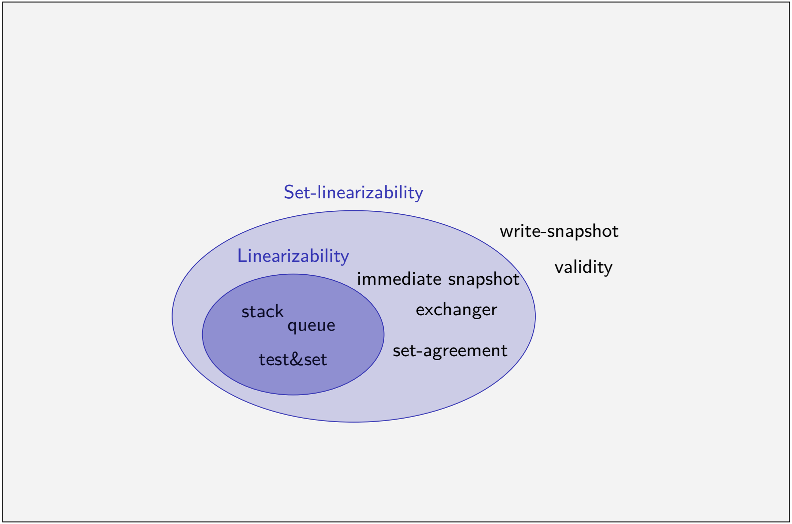

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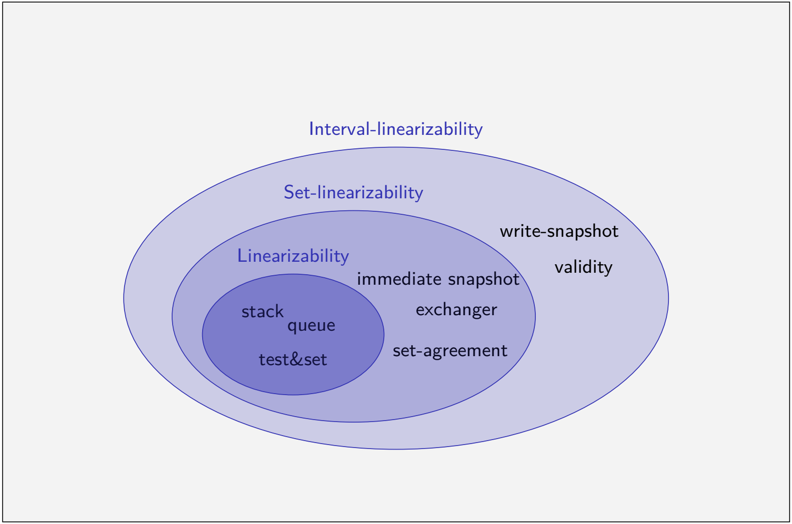

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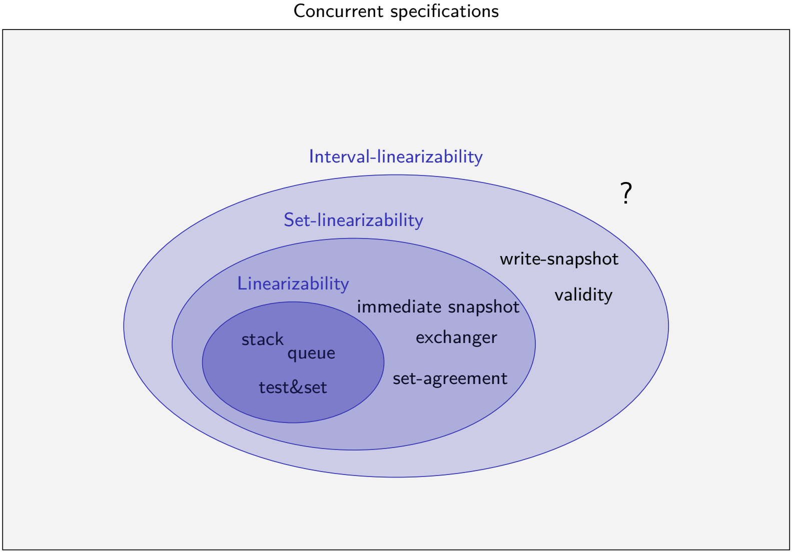
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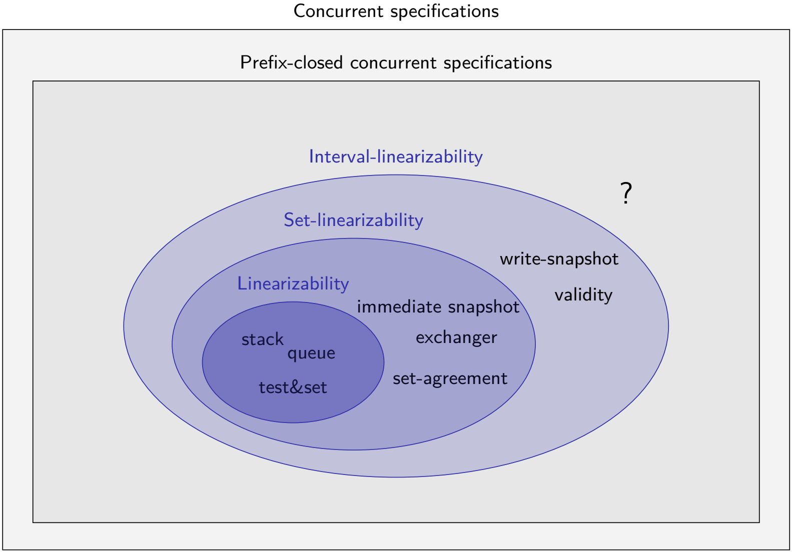
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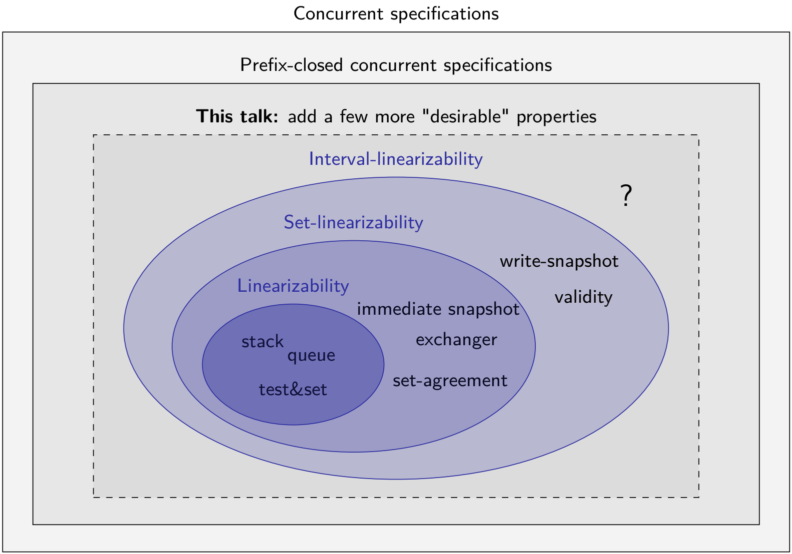

Concurrent specifications
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This talk: add a few more "desirable" properties
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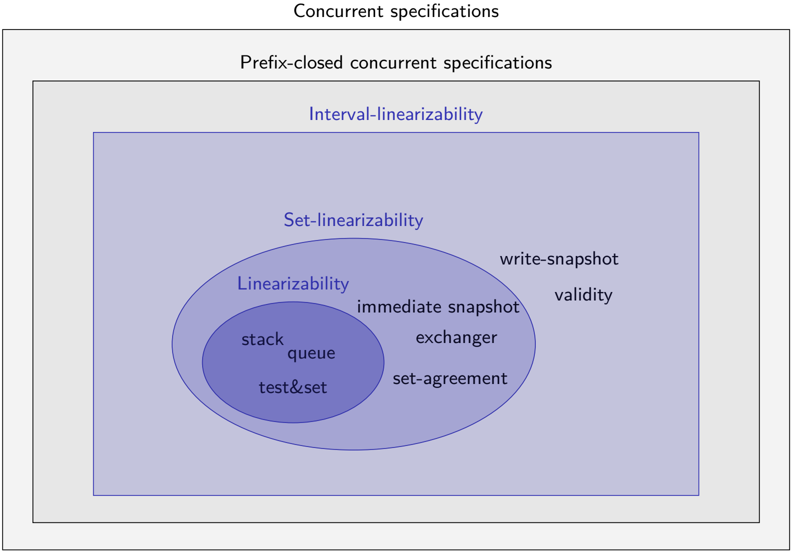
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Relevant concurrent specifications

We write ConcSpec for the set of concurrent specifications $\sigma \subseteq \mathcal{T}$ satisfying the following properties.

(1) *prefix-closure*: if $t \cdot t' \in \sigma$ then $t \in \sigma$,

(2) *non-emptiness*: $\varepsilon \in \sigma$,

(3) *receptivity*: if $t \in \sigma$ and t has no pending invocation of process i, then $t \cdot \mathsf{i}_i^x \in \sigma$ for every input value x,

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(5) σ has the *expansion* property.

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A concurrent specification satisfies the expansion property if:

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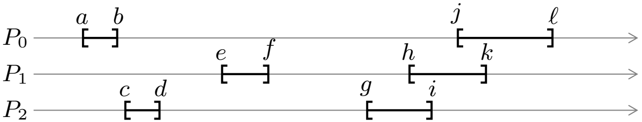

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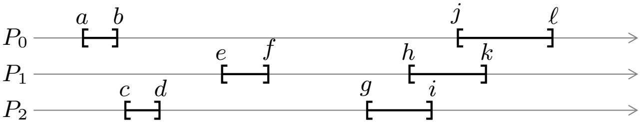
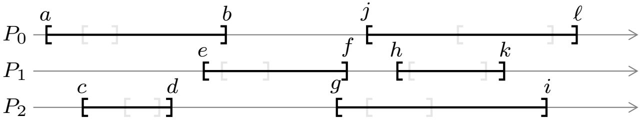


if we *expand* the intervals,



then the resulting trace is still correct.

Example: the Exchanger object

Similar to the one available in Java[1]: *"A synchronization point at which threads can pair and swap elements within pairs".*
Here, we consider a wait-free variant.

[1] `java.util.concurrent.Exchanger<V>`

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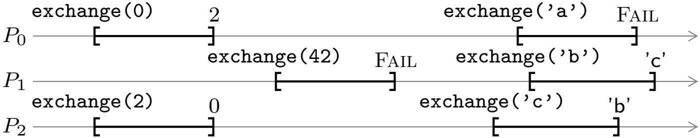
A typical execution of the exchanger looks like this:



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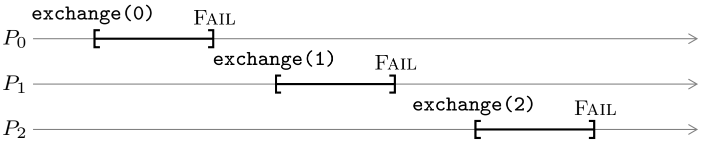
Example: the Exchanger object (2)

The following execution is correct:



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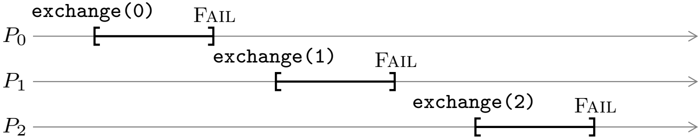
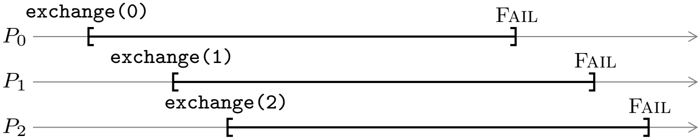


Hence, according to the expansion property,



should be considered correct too!

Linearizability gives expansion for free

Linearizability-based techniques always produce specifications which satisfy the expansion property.

Theorem

For every sequential specification σ, $\mathsf{Lin}(\sigma) \in \mathsf{ConcSpec}$*.*

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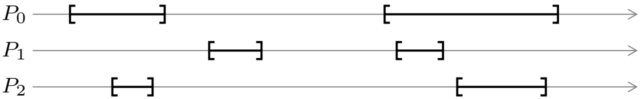

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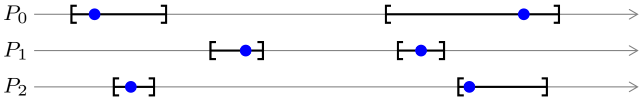

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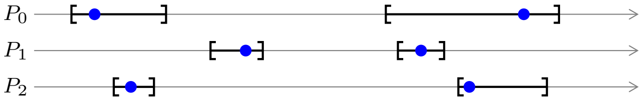
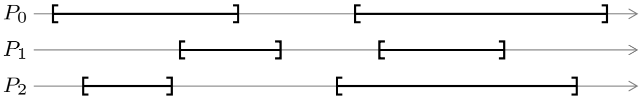


Then any trace obtained by expanding it is still linearizable.



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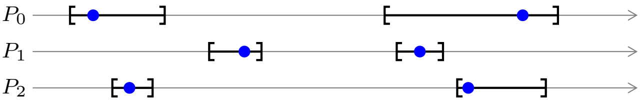

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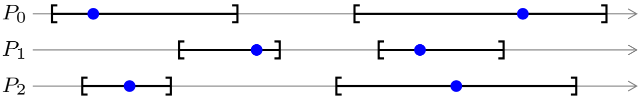

□

A Galois connection

$$\begin{tikzcd} \text{SeqSpec} \arrow[rr, bend right, "\text{Lin}"'] & & \text{ConcSpec} \end{tikzcd}$$

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Theorem

The maps Lin *and* U *form a Galois connection: for every* $\sigma \in \mathsf{SeqSpec}$ *and* $\tau \in \mathsf{ConcSpec}$,

$$\mathsf{Lin}(\sigma) \subseteq \tau \qquad \iff \qquad \sigma \subseteq \mathsf{U}(\tau)$$

Applications

- By the properties of Galois connections,

$$\mathsf{Lin}(\mathsf{U}(\mathsf{Lin}(\sigma))) = \mathsf{Lin}(\sigma)$$

  This yields a simple criterion to check whether a given specification τ is linearizable: check whether $\mathsf{Lin}(\mathsf{U}(\tau)) = \tau$.

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- The Galois connection for interval linearizability has the following corollary:

Theorem

ConcSpec *is the set of interval-linearizable specifications.*

A computational model

We fix a set $\{A_1, \ldots, A_k\}$ of shared objects, along with their concurrent specifications.

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We fix a set $\{A_1, \ldots, A_k\}$ of shared objects, along with their concurrent specifications.

A program P using these objects can:

- call the objects,
- do local computations,
- use branching, loops.

A protocol $\mathcal{P}$ consists of one program for each process.

```
consensus(v) {
  b.write(v);
  x := t.test&set();
  if (x = 0)
    return v;
  else
    v' := a.read();
    return v';
}
```

A computational model (2)

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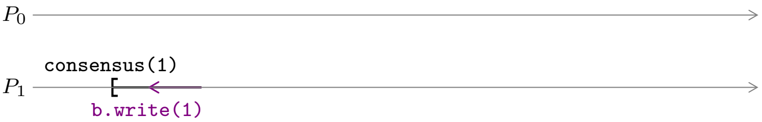

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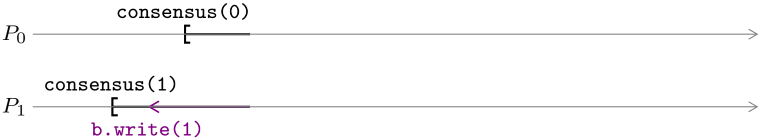

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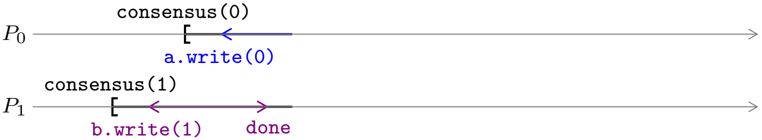

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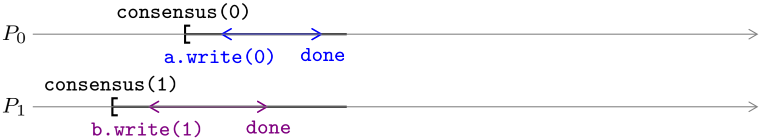

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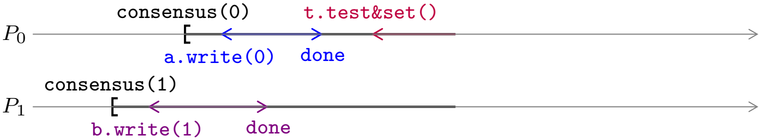

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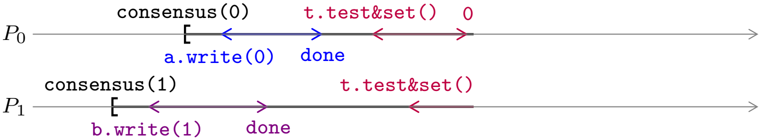

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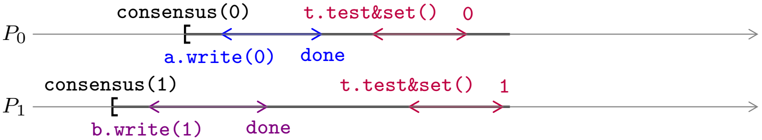

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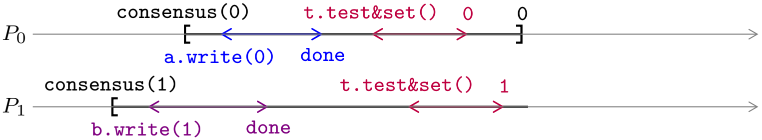

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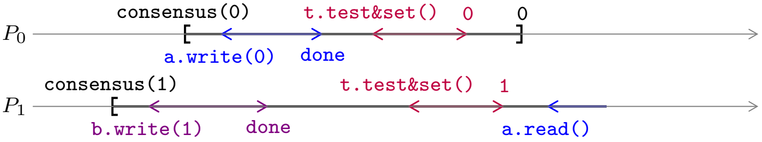

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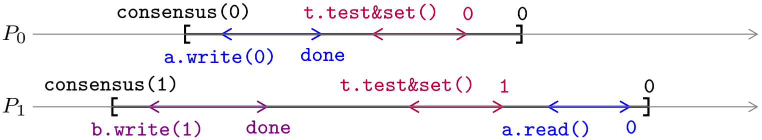

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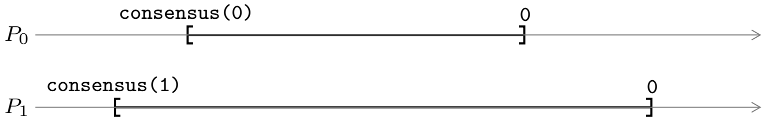

The semantics $[\![\mathcal{P}]\!]$ of a protocol is the set of execution traces that it can produce.

It implements an object specification $S \in \text{ConcSpec}$ if $[\![\mathcal{P}]\!] \subseteq S$.

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Theorem

For any wait-free protocol $\mathcal{P}$, $\quad [\![\mathcal{P}]\!] \in \mathsf{ConcSpec}$.

Asynchronous computability theorem

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Theorem

A wait-free protocol implements a task if and only if there exists a decision map from the protocol complex to the output complex that makes the diagram commute.

Future work

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- Game semantics perspective
  - our notion of implementation looks like the composition of strategies
  - can we characterize the immediate-snapshot strategies, and deduce impossibility results from it?

Thanks!