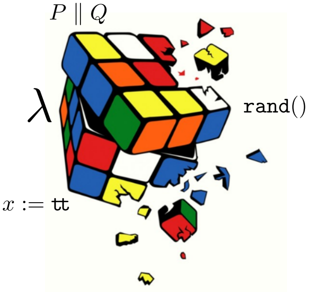
Fix Your Semantic Cube Using This Simple Trick

Pierre Clairambault
CNRS, LIP, ENS Lyon



chocola, 27/09/19.

I. Background : Game Semantics

Game Semantics by Example (Call-By-Name)

A **term**:

$$\lambda f^{\mathbb{U}\to\mathbb{U}}.\,\textbf{newref}\ r\ \textbf{in}\ f\ (r := \mathbf{tt});\ !r \qquad : \qquad (\mathbb{U} \to \mathbb{U}) \to \mathbb{B}$$

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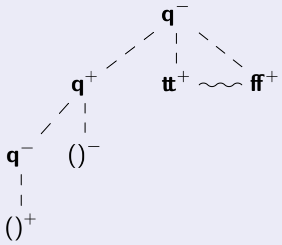
A **game**



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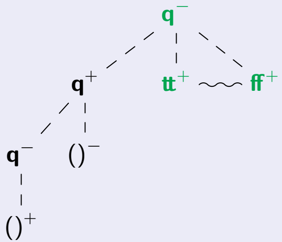

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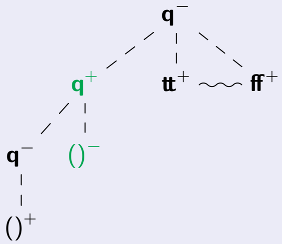

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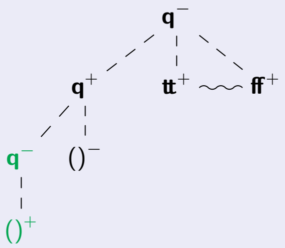

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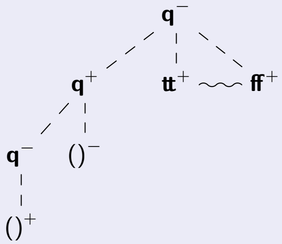

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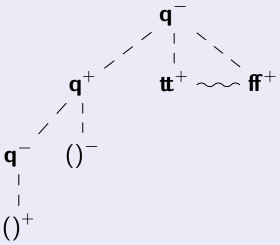
A **play**

$$(\mathbb{U} \quad \to \quad \mathbb{U}) \quad \to \quad \mathbb{B}$$

A **game**



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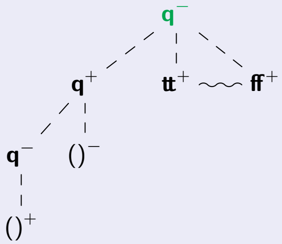
$$\begin{array}{ccccc} (\mathbb{U} & \to & \mathbb{U}) & \to & \mathbb{B} \\ & & & & \mathbf{q}^{-} \end{array}$$

A **game**



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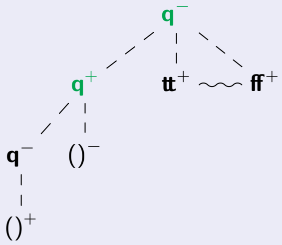
A play

($\mathbb{U}$	$\to$	$\mathbb{U}$)	$\to$	$\mathbb{B}$
				$\mathbf{q}^-$
		$\mathbf{q}^+$		

A game



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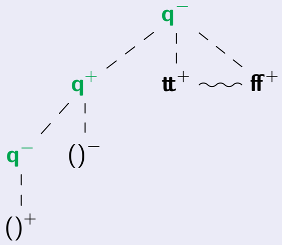
 & & & & \mathbf{q}^- \\
 & & \mathbf{q}^+ & & \\
\mathbf{q}^- & & & &
\end{array}$$

A game



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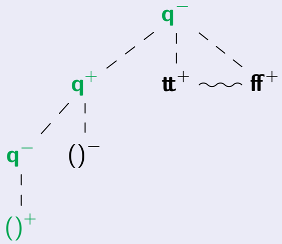
A play

$(\mathbb{U}$	$\to$	$\mathbb{U})$	$\to$	$\mathbb{B}$
				$\mathbf{q}^-$
		$\mathbf{q}^+$		
$\mathbf{q}^-$				
$()^+$				

A game



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 & & \mathbf{q}^{+} & & \\
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()^{+} & & & & \\
 & & ()^{-} & &
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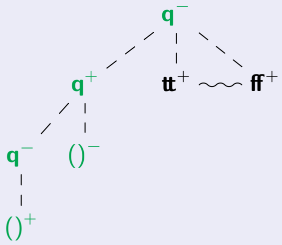
A game



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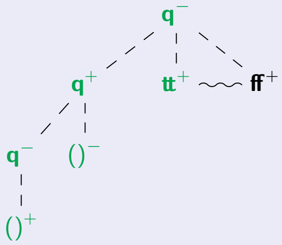
				$\mathbf{q}^-$
		$\mathbf{q}^+$		
$\mathbf{q}^-$				
$()^+$				
		$()^-$		
				$\mathtt{tt}^+$

A game



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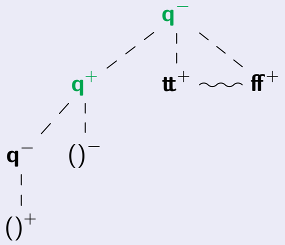

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 & & ()^- & &
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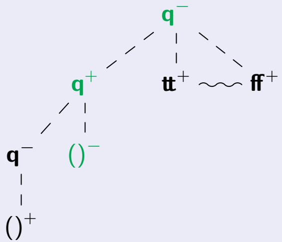
A **game**



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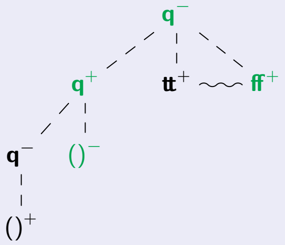
A game



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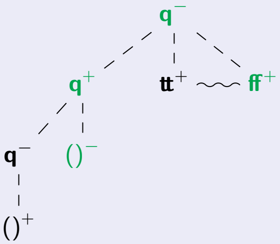

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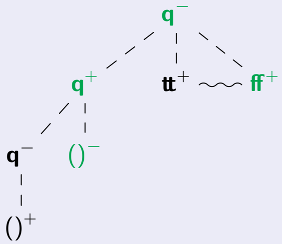
A **game**



The **strategy** interpreting a term is the set of **plays** realized by that term.

Types as Games as Event Structures

Definition

An **event structure** is a tuple $E = \langle |E|, \leq_E, \#_E \rangle$ where:

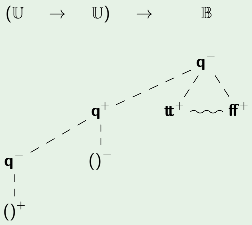
- $|E|$ is a set of **events**,
- $\leq_E$ is a partial order called **causality**,
- $\#_E$ is an irreflexive symmetric binary relation called **conflict**.

satisfying some axioms. A **game** is an event structure A with

$$\mathrm{pol}_A : |A| \to \{-, +\}$$

indicating for each event its **polarity**.

Games as Event Structures



Configurations

Definition

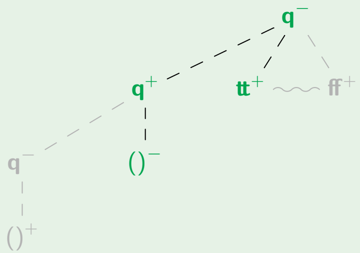
A (finite) **configuration** of an event structure E is a finite set $x \subseteq |E|$ which is:

- **Down-closed**: for all $e \in x$, for all $e' \leq_E e$, we have $e' \in x$;
- **Consistent**: for all $e, e' \in x$, we have $\neg(e \#_E e')$.

The set of (finite) configurations of E is written $\mathcal{C}(E)$.

Configurations

$$(\mathbb{U} \quad \to \quad \mathbb{U}) \quad \to \quad \mathbb{B}$$



Plays

Definition

An **(alternating) play** on game A is a finite sequence of events $a_1 \dots a_n$ such that $\mathrm{pol}_A(a_1) = -$, for all $1 \leq i \leq n$, $\mathrm{pol}_A(a_i) \neq \mathrm{pol}_A(a_{i+1})$ and

$$\{a_1, \dots, a_i\} \in \mathcal{C}(A)\,.$$

We write **AltPlays**(A) the set of (alternating) plays on A.

A **play**

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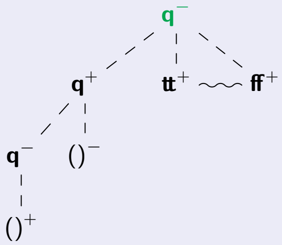

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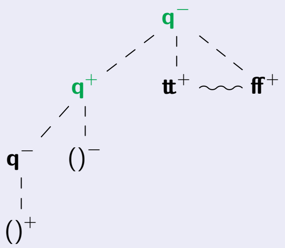

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\end{array}$$

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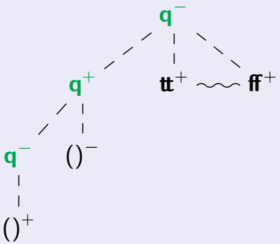

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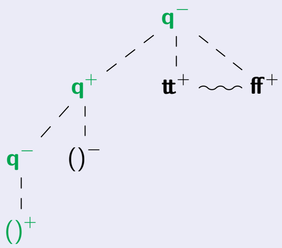

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A **play**

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 & & ()^- & &
\end{array}$$

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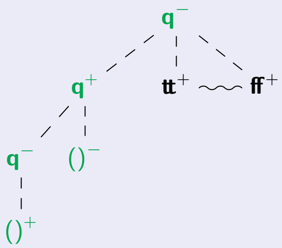

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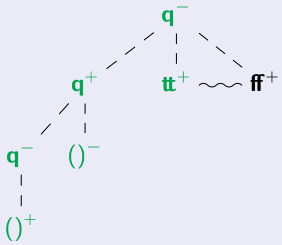
 & & \mathbf{q}^+ & & \\
\mathbf{q}^- & & & & \\
()^+ & & & & \\
 & & ()^- & & \\
 & & & & \mathbf{tt}^+
\end{array}$$

A game



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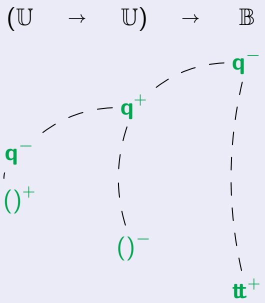

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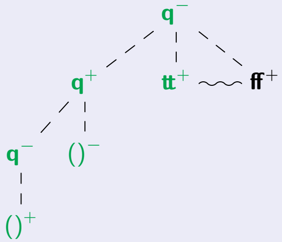
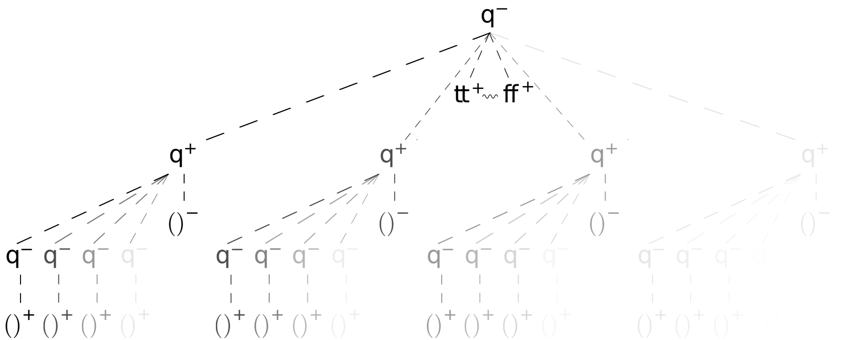


The game for $(\mathbb{U} \to \mathbb{U}) \to \mathbb{B}$ with repetitions



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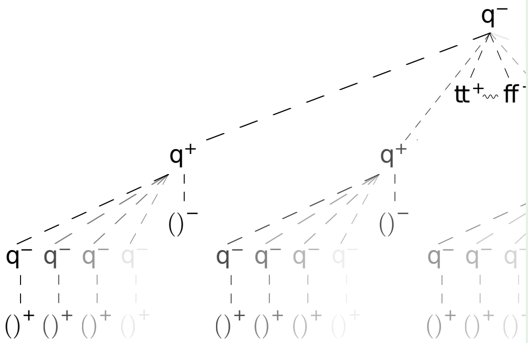
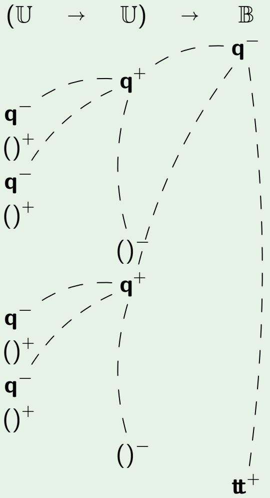


A play with repetitions



Question: is this play realisable?

$$\begin{array}{ccccc}
(\mathbb{U} & \to & \mathbb{U}) & \to & \mathbb{B} \\
 & & & & \mathbf{q}^{-} \\
 & & \mathbf{q}^{+} & & \\
\mathbf{q}^{-} & & & & \\
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\end{array}$$

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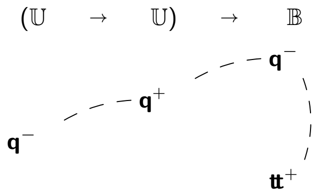
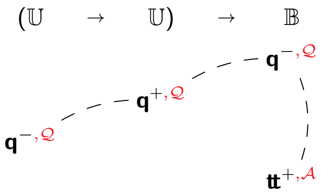


$$\lambda f^{\mathbb{U}\to\mathbb{U}}.\,\mathbf{callcc}\,(\lambda k^{\mathbb{B}\to\mathbb{U}}.\,f\,(k\,\mathbf{tt})) : (\mathbb{U} \to \mathbb{U}) \to \mathbb{B}$$

Question: is this play realisable?



$$\lambda f^{\mathbb{U}\rightarrow\mathbb{U}}.\,\mathbf{callcc}\,(\lambda k^{\mathbb{B}\rightarrow\mathbb{U}}.\,f\;(k\;\mathbf{tt})) : (\mathbb{U} \rightarrow \mathbb{U}) \rightarrow \mathbb{B}$$

Question: is this play realisable?

$$
\begin{array}{ccccc}
(\mathbb{U} & \to & \mathbb{U}) & \to & \mathbb{B} \\
 & & & & \mathbf{q}^{-,\color{red}{\mathcal{Q}}} \\
 & & \mathbf{q}^{+,\color{red}{\mathcal{Q}}} & & \\
\mathbf{q}^{-,\color{red}{\mathcal{Q}}} & & & & \\
 & & & & \mathbf{tt}^{+,\color{red}{\mathcal{A}}}
\end{array}
$$

$$\lambda f^{\mathbb{U}\to\mathbb{U}}.\,\mathbf{callcc}\,(\lambda k^{\mathbb{B}\to\mathbb{U}}.\,f\,(k\,\mathbf{tt})) : (\mathbb{U} \to \mathbb{U}) \to \mathbb{B}$$

Theorem

This play is **non well-bracketed** *and cannot be realized without* **callcc***.*

Question: is this play realisable?

$$\begin{array}{ccccc}
(\mathbb{B} & \to & \mathbb{U}) & \to & \mathbb{U} \\
 & & & & \mathbf{q}^- \\
 & & \mathbf{q}^+ & & \\
\mathbf{q}^- & & & & \\
\mathbf{ff}^+ & & & & \\
\mathbf{q}^- & & & & \\
\mathbf{tt}^+ & & & &
\end{array}$$

Question: is this play realisable?

$$
\begin{array}{ccccc}
(\mathbb{B} & \rightarrow & \mathbb{U}) & \rightarrow & \mathbb{U} \\
 & & & & \mathbf{q}^{-} \\
 & & \mathbf{q}^{+} & & \\
\mathbf{q}^{-} & & & & \\
\mathbf{ff}^{+} & & & & \\
\mathbf{q}^{-} & & & & \\
\mathbf{tt}^{+} & & & &
\end{array}
$$

$$\lambda f^{\mathbb{B}\rightarrow\mathbb{U}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (\mathbf{let}\ x = !r\ \mathbf{in}\ r := \mathbf{tt};\ x) : (\mathbb{B} \rightarrow \mathbb{U}) \rightarrow \mathbb{U}$$

Question: is this play realisable?

$$
\begin{array}{ccccc}
(\mathbb{B} & \to & \mathbb{U}) & \to & \mathbb{U} \\
 & & & & \mathbf{q}^{-} \\
 & & \mathbf{q}^{+} & & \\
\mathbf{q}^{-} & & & & \\
\mathbf{ff}^{+} & & & & \\
\mathbf{q}^{-} & & & & \\
\mathbf{tt}^{+} & & & &
\end{array}
$$

$$\lambda f^{\mathbb{B}\to\mathbb{U}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (\mathbf{let}\ x = !r\ \mathbf{in}\ r := \mathbf{tt};\ x) : (\mathbb{B} \to \mathbb{U}) \to \mathbb{U}$$

Theorem

This play is **non-innocent** *and cannot be realized without* **references**.

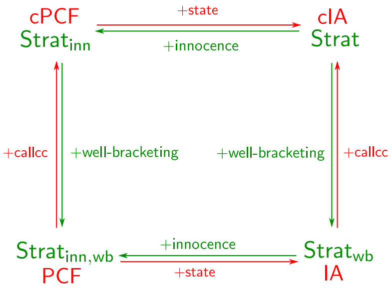
Full abstraction results

$$
\begin{tikzcd}
\mathrm{cPCF} \arrow[r, "+\mathrm{state}"'] & \mathrm{cIA} \arrow[rr, bend left, "\llbracket - \rrbracket"] & & \mathrm{Strat}_{\mathrm{inn}} \arrow[d, "+\text{well-bracketing}"] & \mathrm{Strat} \arrow[l, "+\mathrm{innocence}"] \arrow[d, "+\text{well-bracketing}"] \\
\mathrm{PCF} \arrow[u, "+\mathrm{callcc}"] \arrow[r, "+\mathrm{state}"] & \mathrm{IA} \arrow[u, "+\mathrm{callcc}"'] & & \mathrm{Strat}_{\mathrm{inn,wb}} & \mathrm{Strat}_{\mathrm{wb}} \arrow[l, "+\mathrm{innocence}"']
\end{tikzcd}
$$

all correspondences being **fully abstract** or **intensionally fully abstract**.[1]

[1] Follows from work in the late 90s from Abramsky, Hyland, Laird, McCusker, Ong.

Orthogonality of control and state



Theorem

Suppose a program M in **cIA** *is observationally equivalent to*

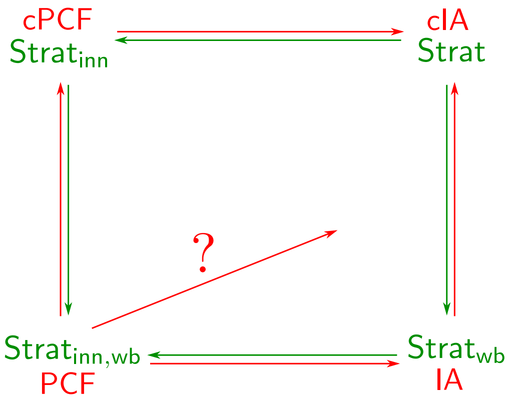
- *A program M_1 that does not use* **callcc***;*
- *A program M_2 that does not use* **references***.*

Then, M is observationally equivalent to M' in pure **PCF***.*

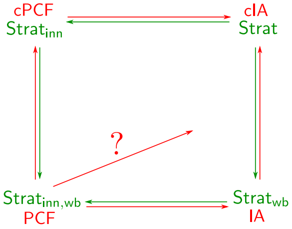
The “semantic cube”

$$
\begin{tikzcd}
\begin{matrix}\mathsf{cPCF}\\ \mathsf{Strat}_{\mathsf{inn}}\end{matrix} \arrow[r, shift left] \arrow[d, shift left] & \begin{matrix}\mathsf{cIA}\\ \mathsf{Strat}\end{matrix} \arrow[l, shift left] \arrow[d, shift right] \\
\begin{matrix}\mathsf{Strat}_{\mathsf{inn,wb}}\\ \mathsf{PCF}\end{matrix} \arrow[u, shift left] \arrow[r, shift right] & \begin{matrix}\mathsf{Strat}_{\mathsf{wb}}\\ \mathsf{IA}\end{matrix} \arrow[l, shift right] \arrow[u, shift right]
\end{tikzcd}
$$

The "semantic cube"



The "semantic cube"

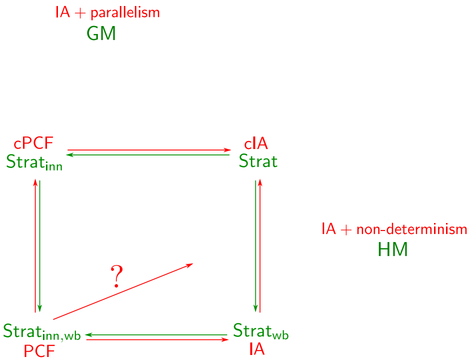


The “semantic cube”

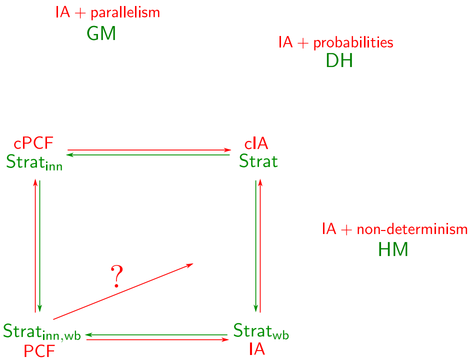
$$\begin{array}{c}\text{IA} + \text{parallelism}\\ \mathsf{GM}\end{array}$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{c}\text{cPCF}\\ \mathsf{Strat}_{\mathsf{inn}}\end{array} \arrow[r, shift left] \arrow[d, shift left] & \begin{array}{c}\text{cIA}\\ \mathsf{Strat}\end{array} \arrow[l, shift left] \arrow[d, shift right] \\
\begin{array}{c}\mathsf{Strat}_{\mathsf{inn,wb}}\\ \text{PCF}\end{array} \arrow[u, shift left] \arrow[r, shift right] \arrow[ur, phantom, "?"] & \begin{array}{c}\mathsf{Strat}_{\mathsf{wb}}\\ \text{IA}\end{array} \arrow[l, shift right] \arrow[u, shift right]
\end{tikzcd}$$

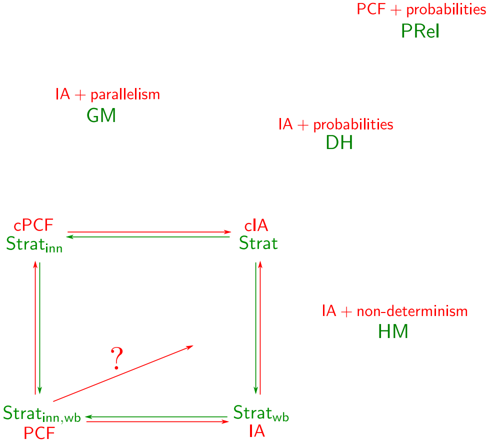
The "semantic cube"



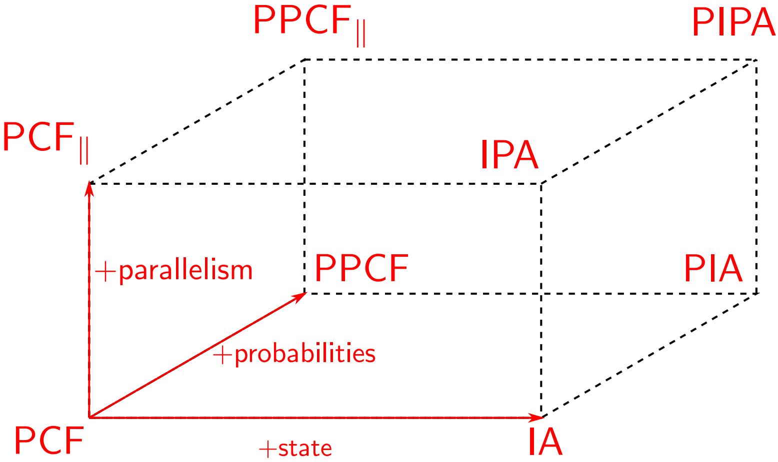
The “semantic cube”



The “semantic cube”

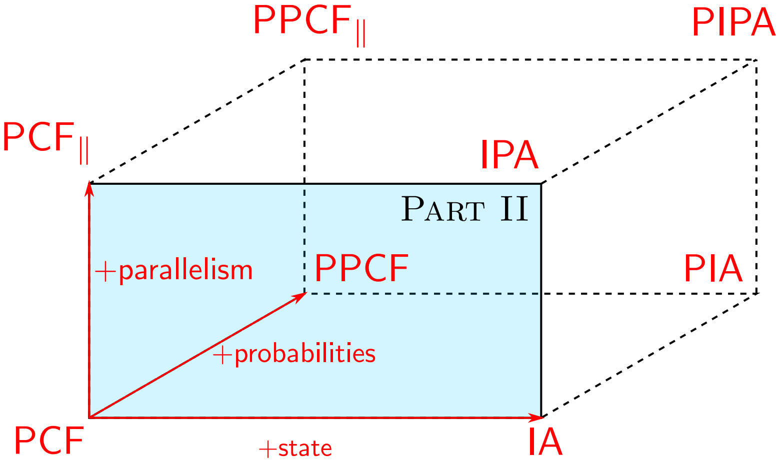


Outline

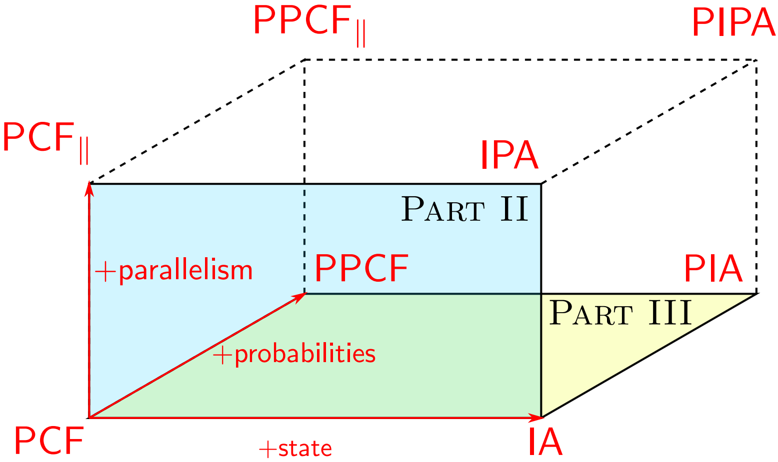


$PPCF_{\parallel}$
PIPA
$PCF_{\parallel}$
IPA
+parallelism
PPCF
PIA
+probabilities
PCF
+state
IA

Outline


PPCF$_\|$
PIPA
PCF$_\|$
IPA
Part II
+parallelism
PPCF
PIA
+probabilities
PCF
+state
IA

Outline



PPCF$_{\|}$
PIPA
PCF$_{\|}$
IPA
Part II
+parallelism
PPCF
PIA
Part III
+probabilities
PCF
+state
IA

II. Concurrent Games and Parallel Innocence

IPA and its components

Types.

$$\begin{array}{lll} A, B & ::= & \mathbb{U} \mid \mathbb{B} \mid \mathbb{N} \mid A \to B \quad \textbf{PCF} \\ & & \mid \textbf{ref} \qquad\qquad\qquad\quad\ \ \textbf{+state} \end{array}$$

Terms.

$$\begin{array}{llll} M, N & ::= & x \mid M\,N \mid \lambda x.\,M \mid Y & \lambda Y\textbf{-calculus} \\ \\ & & \mid \textbf{tt} \mid \textbf{ff} \mid \textbf{if}\ M\ N_1\ N_2 & \\ & & \mid n \mid \textbf{succ}\ M \mid \textbf{pred}\ M \mid \textbf{iszero}\ M & \\ & & \mid \textbf{skip} \mid M; N & \textbf{PCF} \\ \\ & & \mid \textbf{newref}\ v := b\,\textbf{in}\ M \mid M := N \mid\ !M & +\textbf{state} \\ \\ & & \mid \textbf{let} \begin{pmatrix} x & = & M \\ y & = & N \end{pmatrix} \textbf{in}\ T & +\textbf{parallel} \end{array}$$

$\hookrightarrow$ **PCF + state + parallel = IPA**

Standard typing rules and call-by-name operational semantics.

Roadmap

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\underset{?}{\mathsf{PCF}_{\parallel}} \arrow[r, shift left, "+\text{state}"] \arrow[d, shift left, "+\text{sequentiality}"] & \underset{?}{\mathsf{IPA}} \arrow[l, shift left, "+\text{parallel innocence}"] \arrow[d, shift left, "+\text{sequentiality}"] \\
\underset{?}{\mathsf{PCF}} \arrow[u, shift left, "+\text{parallelism}"] \arrow[r, shift left, "+\text{state}"] & \underset{?}{\mathsf{IA}} \arrow[u, shift left, "+\text{parallelism}"] \arrow[l, shift left, "+\text{parallel innocence}"]
\end{tikzcd}
$$

Roadmap

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{c}\mathsf{PCF}_{\parallel}\\ ?\end{array} \arrow[r, shift left, "+\text{state}"] \arrow[d, shift left, "+\text{sequentiality}"] & \begin{array}{c}\mathsf{IPA}\\ ?\end{array} \arrow[l, shift left, "+\text{parallel innocence}"] \arrow[d, shift left, "+\text{sequentiality}"] \\
\begin{array}{c}\mathsf{PCF}\\ ?\end{array} \arrow[u, shift left, "+\text{parallelism}"] \arrow[r, shift left, "+\text{state}"] & \begin{array}{c}\mathsf{IA}\\ ?\end{array} \arrow[l, shift left, "+\text{parallel innocence}"] \arrow[u, shift left, "+\text{parallelism}"]
\end{tikzcd}
$$

Non-alternating game semantics for IPA [2]

Theorem

The model **GM** *of games and well-bracketed non-alternating strategies is* **fully abstract for IPA**.

Definition

An **(non-alternating) play** on game A is a finite sequence of events $a_1 \dots a_n$ such that for all $1 \leq i \leq n$,

$$\{a_1, \dots, a_i\} \in \mathcal{C}(A)\,.$$

We write $\mathbf{Plays}(A)$ the set of (non-alternating) plays on A.

Definition

A **non-alternating strategy** $\sigma : A$ is a subset

$$\sigma \subseteq \mathbf{Plays}(A)$$

satisfying some conditions.

[2] D. Ghica, A. Murawski. *Angelic Semantics of Fine-Grained Concurrency*, FoSSaCS 2004.

Non-alternating plays

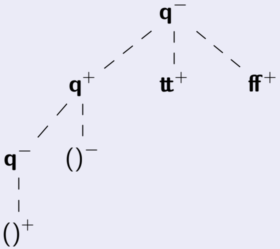
A **term**:

$$\lambda f^{\mathbb{U}\to\mathbb{U}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \qquad : \qquad (\mathbb{U} \to \mathbb{U}) \to \mathbb{B}$$

A **play**

$$(\mathbb{U} \quad \to \quad \mathbb{U}) \quad \to \quad \mathbb{B}$$

A **game**



Non-alternating plays

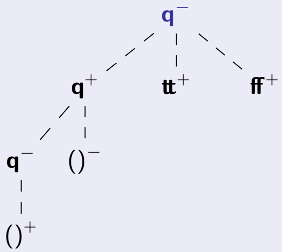
A **term**:

$$\lambda f^{\mathbb{U}\to\mathbb{U}}.\,\textbf{newref}\ r\ \textbf{in}\ f\ (r := \mathbf{tt});\ !r \qquad : \qquad (\mathbb{U}\to\mathbb{U})\to\mathbb{B}$$

A **play**

$$\begin{array}{ccccc} (\mathbb{U} & \to & \mathbb{U}) & \to & \mathbb{B} \\ & & & & \mathbf{q}^- \end{array}$$

A **game**



Non-alternating plays

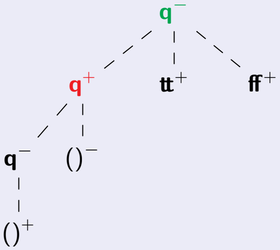
A **term**:

$$\lambda f^{\mathbb{U}\to\mathbb{U}}.\, \textbf{newref}\ r\ \textbf{in}\ f\ (r := \mathbf{tt});\ !r \qquad : \qquad (\mathbb{U} \to \mathbb{U}) \to \mathbb{B}$$

A **play**

$$\begin{array}{ccccc}
(\mathbb{U} & \to & \mathbb{U}) & \to & \mathbb{B} \\
 & & & & \mathbf{q}^{-} \\
 & & \mathbf{q}^{+} & &
\end{array}$$

A **game**



Non-alternating plays

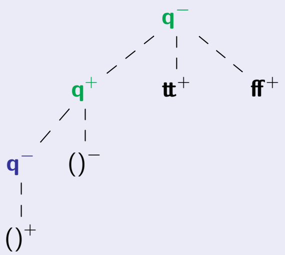
A **term**:

$$\lambda f^{\mathbb{U}\to\mathbb{U}}.\, \mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \qquad : \qquad (\mathbb{U} \to \mathbb{U}) \to \mathbb{B}$$

A play

$$\begin{array}{ccccc}
(\mathbb{U} & \to & \mathbb{U}) & \to & \mathbb{B} \\
 & & & & \mathbf{q}^{-} \\
 & & \mathbf{q}^{+} & & \\
\mathbf{q}^{-} & & & &
\end{array}$$

A game



Non-alternating plays

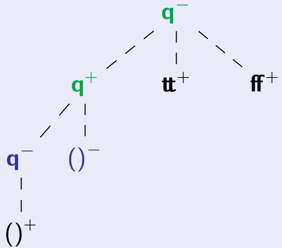
A **term**:

$$\lambda f^{\mathbb{U}\to\mathbb{U}}.\,\textbf{newref}\ r\ \textbf{in}\ f\ (r := \mathtt{tt});\ !r \qquad : \qquad (\mathbb{U} \to \mathbb{U}) \to \mathbb{B}$$

A play

$(\mathbb{U}$	$\to$	$\mathbb{U})$	$\to$	$\mathbb{B}$
				$\mathbf{q}^-$
		$\mathbf{q}^+$		
$\mathbf{q}^-$				
		$()^-$		

A game



Non-alternating plays

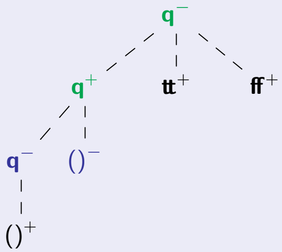
A **term**:

$$\lambda f^{\mathbb{U}\to\mathbb{U}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \qquad : \qquad (\mathbb{U} \to \mathbb{U}) \to \mathbb{B}$$

A play

$$\begin{array}{ccccc}
(\mathbb{U} & \to & \mathbb{U}) & \to & \mathbb{B} \\
 & & & & \mathbf{q}^- \\
 & & \mathbf{q}^+ & & \\
 & & ()^- & & \\
\mathbf{q}^- & & & &
\end{array}$$

A game



Non-alternating plays

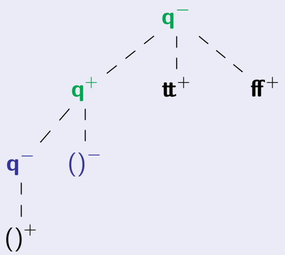
A **term**:

$$\lambda f^{\mathbb{U}\to\mathbb{U}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \qquad : \qquad (\mathbb{U} \to \mathbb{U}) \to \mathbb{B}$$

A play

$$\begin{array}{ccccc}
(\mathbb{U} & \to & \mathbb{U}) & \to & \mathbb{B} \\
 & & & & \mathbf{q}^{-} \\
 & & \mathbf{q}^{+} & & \\
\mathbf{q}^{-} & & & & \\
 & & ()^{-} & &
\end{array}$$

A game



Non-alternating plays

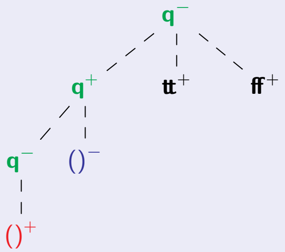
A **term**:

$$\lambda f^{\mathbb{U}\to\mathbb{U}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \qquad : \qquad (\mathbb{U}\to\mathbb{U})\to\mathbb{B}$$

A play

$$\begin{array}{ccccc}
(\mathbb{U} & \to & \mathbb{U}) & \to & \mathbb{B} \\
 & & & & \mathbf{q}^{-} \\
 & & \mathbf{q}^{+} & & \\
\mathbf{q}^{-} & & & & \\
 & & ()^{-} & & \\
()^{+} & & & &
\end{array}$$

A game



Non-alternating plays

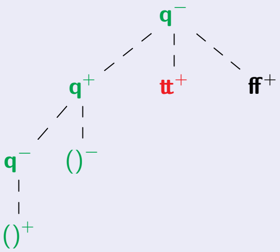
A **term**:

$$\lambda f^{\mathbb{U}\to\mathbb{U}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathbf{tt});\ !r \qquad : \qquad (\mathbb{U} \to \mathbb{U}) \to \mathbb{B}$$

A **play**

$$\begin{array}{ccccc}
(\mathbb{U} & \to & \mathbb{U}) & \to & \mathbb{B} \\
 & & & & \mathbf{q}^{-} \\
 & & \mathbf{q}^{+} & & \\
\mathbf{q}^{-} & & & & \\
 & & ()^{-} & & \\
()^{+} & & & & \\
 & & & & \mathbf{tt}^{+}
\end{array}$$

A **game**



Non-alternating plays

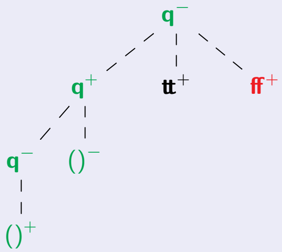
A **term**:

$$\lambda f^{\mathbb{U}\to\mathbb{U}}.\,\textbf{newref}\ r\ \textbf{in}\ f\ (r := \mathbf{tt});\ !r \qquad : \qquad (\mathbb{U} \to \mathbb{U}) \to \mathbb{B}$$

A **play**

$$\begin{array}{ccccc}
(\mathbb{U} & \to & \mathbb{U}) & \to & \mathbb{B} \\
 & & & & \mathbf{q}^- \\
 & & \mathbf{q}^+ & & \\
\mathbf{q}^- & & & & \\
 & & ()^- & & \\
()^+ & & & & \\
 & & & & \mathbf{ff}^+
\end{array}$$

A **game**



Non-alternating plays

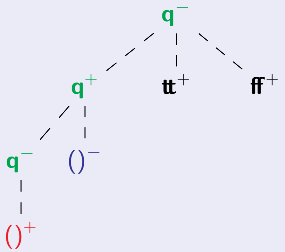
A **term**:

$$\lambda f^{\mathbb{U}\to\mathbb{U}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \quad : \quad (\mathbb{U} \to \mathbb{U}) \to \mathbb{B}$$

A play

$(\mathbb{U}$	$\to$	$\mathbb{U})$	$\to$	$\mathbb{B}$
				$\mathbf{q}^-$
		$\mathbf{q}^+$		
$\mathbf{q}^-$				
		$()^-$		
$()^+$				

A game



Non-alternating plays

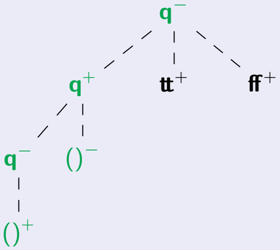
A **term**:

$$\lambda f^{\mathbb{U}\to\mathbb{U}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \qquad : \qquad (\mathbb{U} \to \mathbb{U}) \to \mathbb{B}$$

A **play**

$$\begin{array}{ccccc}
(\mathbb{U} & \to & \mathbb{U}) & \to & \mathbb{B} \\
 & & & & \mathbf{q}^{-} \\
 & & \mathbf{q}^{+} & & \\
\mathbf{q}^{-} & & & & \\
()^{+} & & & & \\
 & & ()^{-} & &
\end{array}$$

A **game**



Non-alternating plays

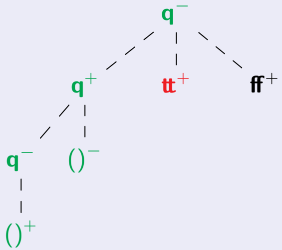
A **term**:

$$\lambda f^{\mathbb{U}\to\mathbb{U}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \qquad : \qquad (\mathbb{U}\to\mathbb{U})\to\mathbb{B}$$

A **play**

$$\begin{array}{ccccc}
(\mathbb{U} & \to & \mathbb{U}) & \to & \mathbb{B} \\
 & & & & \mathbf{q}^{-} \\
 & & \mathbf{q}^{+} & & \\
\mathbf{q}^{-} & & & & \\
()^{+} & & & & \\
 & & ()^{-} & & \\
 & & & & \mathtt{tt}^{+}
\end{array}$$

A **game**



Question: can a program without state realize these two plays?

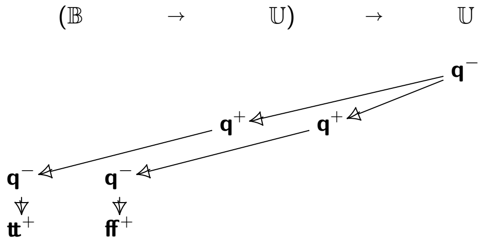
$$
\begin{array}{ccccc}
(\mathbb{B} & \to & \mathbb{U}) & \to & \mathbb{U} \\
 & & & & \mathbf{q}^- \\
 & & \mathbf{q}^+ & & \\
\mathbf{q}^- & & & & \\
\mathbf{ff}^+ & & & &
\end{array}
\qquad\qquad
\begin{array}{ccccc}
(\mathbb{B} & \to & \mathbb{U}) & \to & \mathbb{U} \\
 & & & & \mathbf{q}^- \\
 & & \mathbf{q}^+ & & \\
\mathbf{q}^- & & & & \\
\mathbf{tt}^+ & & & &
\end{array}
$$

Question: can a program without state realize these two plays?

$$\begin{array}{ccccc} (\mathbb{B} & \to & \mathbb{U}) & \to & \mathbb{U} \\ &&&& \mathbf{q}^- \\ && \mathbf{q}^+ && \\ \mathbf{q}^- &&&& \\ \mathbf{ff}^+ &&&& \end{array} \qquad \begin{array}{ccccc} (\mathbb{B} & \to & \mathbb{U}) & \to & \mathbb{U} \\ &&&& \mathbf{q}^- \\ && \mathbf{q}^+ && \\ \mathbf{q}^- &&&& \\ \mathbf{tt}^+ &&&& \end{array}$$

$$\lambda f^{\mathbb{B}\to\mathbb{U}}.\, \mathbf{let} \left(\begin{array}{ccc} x & = & f\,\mathbf{tt} \\ y & = & f\,\mathbf{ff} \end{array}\right) \mathbf{in}\; x;\ y : (\mathbb{B}\to\mathbb{U})\to\mathbb{U}$$

Question: can a program without state realize these two plays?



$$\lambda f^{\mathbb{B}\to\mathbb{U}}.\,\mathbf{let}\ \begin{pmatrix} x & = & f\ \mathbf{tt} \\ y & = & f\ \mathbf{ff} \end{pmatrix}\ \mathbf{in}\ x;\ y : (\mathbb{B}\to\mathbb{U})\to\mathbb{U}$$

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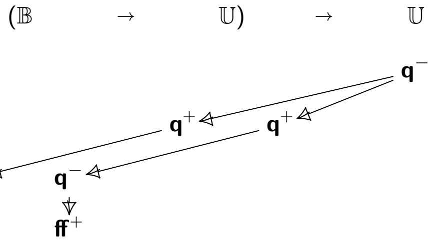

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$\hookrightarrow$ concurrent games[3]

[3] Family of models initiated by Abramsky and Melliès (1999), then Melliès, Mimram, Faggian, Piccolo (2000s), then Rideau, Winskel, Castellan, C., Paquet, Alcolei, de Visme etc. . . (2010s).

Roadmap

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{c}\mathsf{PCF}_{\parallel}\\ ?\end{array} \arrow[r, shift left, "\text{+state}"] \arrow[d, shift left, "\text{+sequentiality}"] & \begin{array}{c}\mathsf{IPA}\\ \mathsf{CG}\end{array} \arrow[l, shift left, "\text{+parallel innocence}"] \arrow[d, shift left, "\text{+sequentiality}"] \\
\begin{array}{c}\mathsf{PCF}\\ ?\end{array} \arrow[u, shift left, "\text{+parallelism}"] \arrow[r, shift left, "\text{+state}"] & \begin{array}{c}\mathsf{IA}\\ ?\end{array} \arrow[l, shift left, "\text{+parallel innocence}"] \arrow[u, shift left, "\text{+parallelism}"]
\end{tikzcd}
$$

Partially ordered plays: augmented configurations

Definition

An **augmentation** on A is a **conflict-free event structure** $\mathbf{q} = \langle |\mathbf{q}|, \leq_{\mathbf{q}} \rangle$ where

$$\mathcal{C}(\mathbf{q}) \subseteq \mathcal{C}(A)\,.$$

An augmentation

$$\begin{tikzcd}
(\mathbb{B} & \to \quad \mathbb{U}) & \to \quad \mathbb{U} \\
 & & \mathbf{q}^- \arrow[dl] \\
 & \mathbf{q}^+ \arrow[dl] \arrow[d] & \\
\mathbf{q}^- \arrow[d] & ()^- \arrow[dr] & \\
\mathbf{tt}^+ & & ()^+
\end{tikzcd}$$

($\rightarrowtriangle$ is the **immediate causality relation**).

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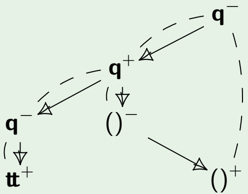
An augmentation

$$(\mathbb{B} \quad \to \quad \mathbb{U}) \quad \to \quad \mathbb{U}$$



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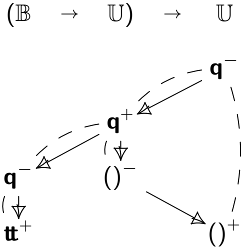
Question: is this augmentation realizable?



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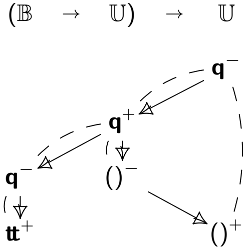

$$\lambda f^{\mathbb{B} \to \mathbb{U}}.\, f\ \mathbf{tt} : (\mathbb{B} \to \mathbb{U}) \to \mathbb{U}$$

Question: is this augmentation realizable?

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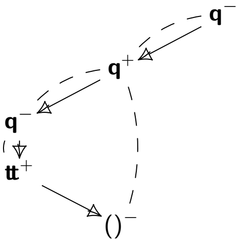

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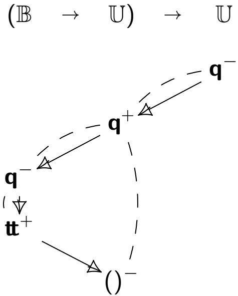
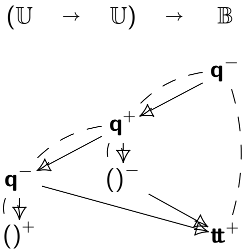


Definition

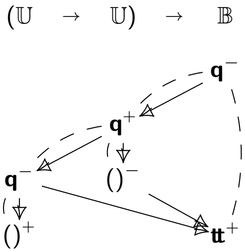
An augmentation $\mathbf{q}$ on A is **courteous** iff for all $a_1 \rightarrowtriangle_{\mathbf{q}} a_2$ such that $\neg(a_1 \rightarrowtriangle_{\mathbf{q}} a_2)$, we have $\mathrm{pol}_A(a_1) = -$ and $\mathrm{pol}_A(a_2) = +$.

We write **Aug**(A) for the set of **courteous augmentations** on A.

Question: is this augmentation realizable?

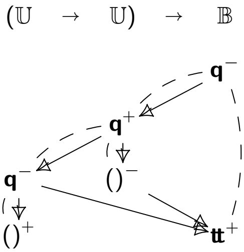


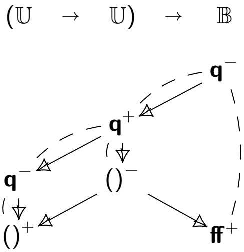
Question: is this augmentation realizable?



$$\lambda f^{\mathbb{U}\to\mathbb{U}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathbf{tt});\ !r \qquad : \qquad (\mathbb{U} \to \mathbb{U}) \to \mathbb{B}$$

Question: is this augmentation realizable?





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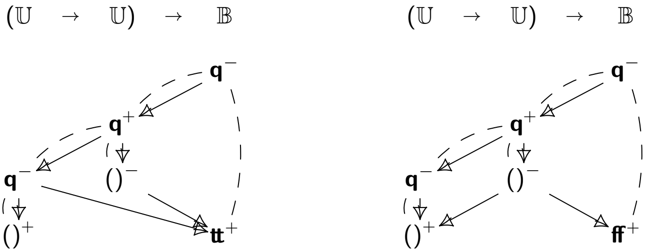


Definition

A **(concurrent) strategy** $\sigma : A$ is a **non-empty, prefix-closed** subset

$$\sigma \subseteq \mathbf{Aug}(A)$$

closed under extensions by Opponent events.

Causal intensional full abstraction for IPA [4]

Theorem

The model **CG** *of* **games** *and* **(well-bracketed) concurrent strategies** *is intensionally fully abstract for IPA.*

Proof.

If $\sigma : A$ is a strategy, then

$$\mathbf{Plays}(\sigma) = \cup\{\mathbf{Plays}(\mathbf{q}) \mid \mathbf{q} \in \sigma\}$$

is a strategy in the Ghica-Murawski sense.

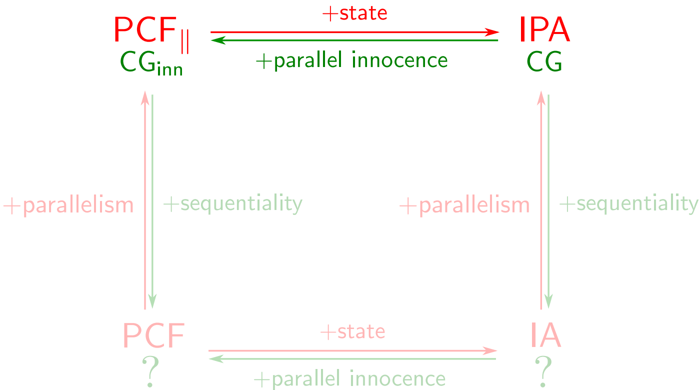
This forms a functor

$$\mathbf{Plays}(-) : \mathbf{CG} \to \mathbf{GM}$$

preserving the interpretation. $\square$

[4] S. Castellan, P.C. *Causality vs. interleavings in concurrent game semantics*, CONCUR 2016.

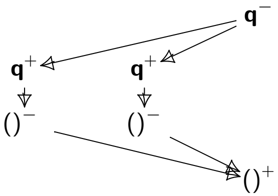
Roadmap



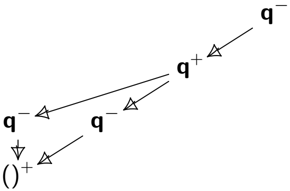
$\mathrm{PCF}_{\parallel}$
$\mathrm{CG}_{\mathrm{inn}}$
+state
+parallel innocence
IPA
CG
+parallelism
+sequentiality
+parallelism
+sequentiality
PCF
?
+state
+parallel innocence
IA
?

Question: which of these two is realizable only with state?

$\mathbb{U} \rightarrow \mathbb{U} \rightarrow \mathbb{U}$



$(\mathbb{U} \rightarrow \mathbb{U} \rightarrow \mathbb{U}) \rightarrow \mathbb{U}$



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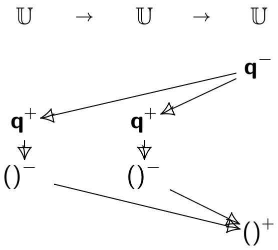
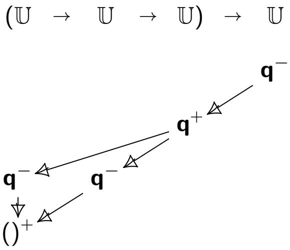
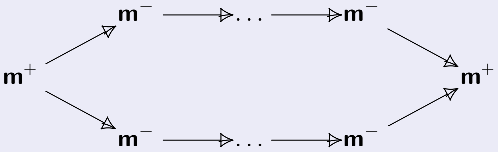




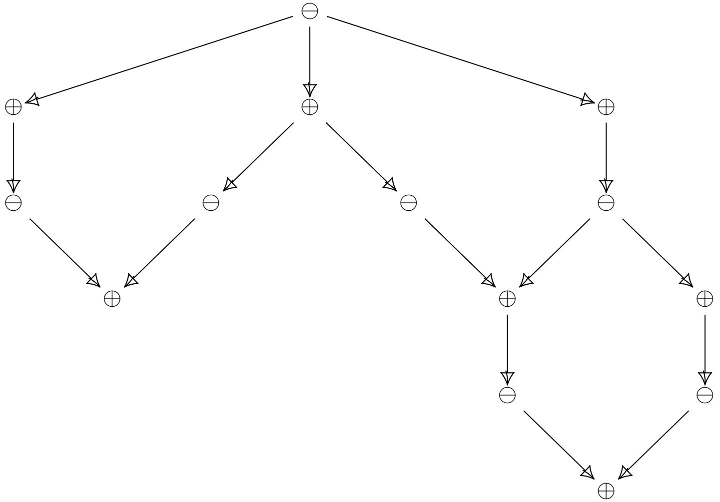
Definition

An augmentation $\mathbf{q} \in \mathbf{Aug}(A)$ is **innocent** if it has **no pattern of the form**



A strategy $\sigma : A$ is **innocent** if any $\mathbf{q} \in \sigma$ is.

The causal shape of parallel innocence



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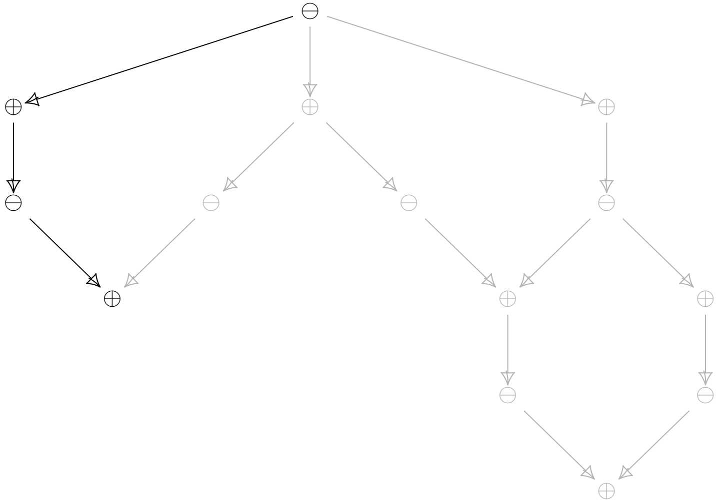

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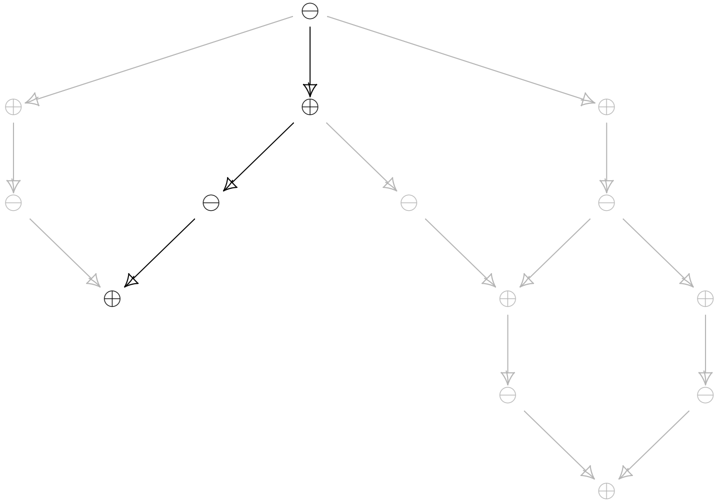

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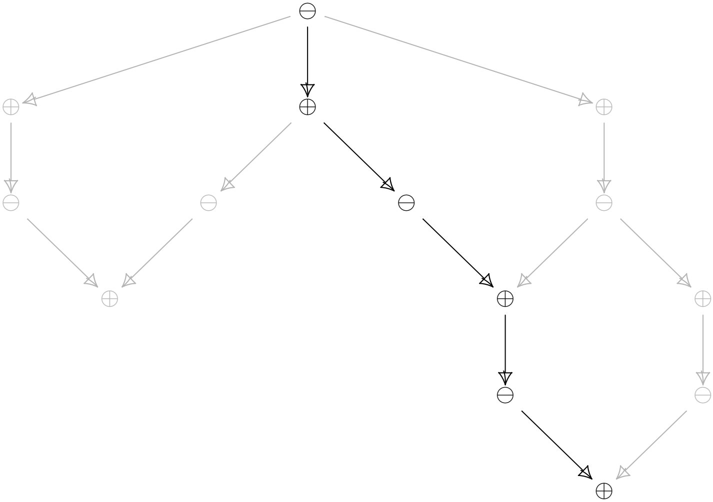

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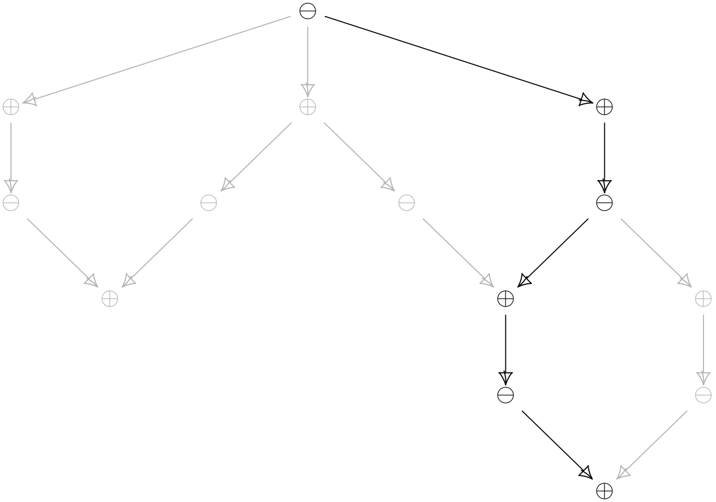

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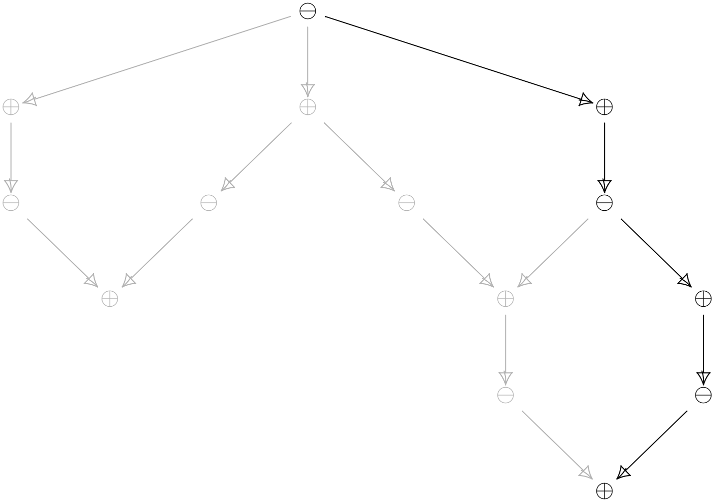
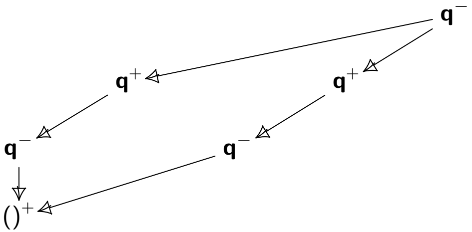


Question: is the following augmentation realizable without state?

$$(\mathbb{U} \to \mathbb{U}) \to (\mathbb{U} \to \mathbb{U}) \to \mathbb{U}$$



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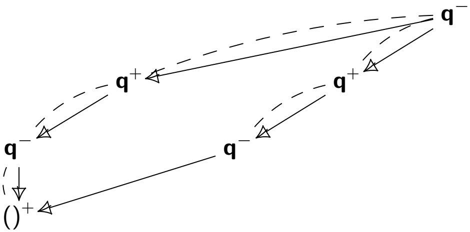

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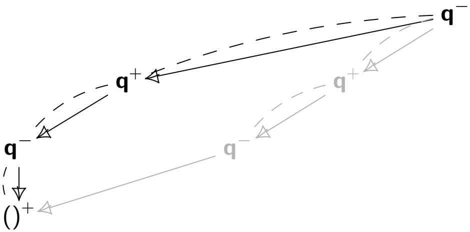

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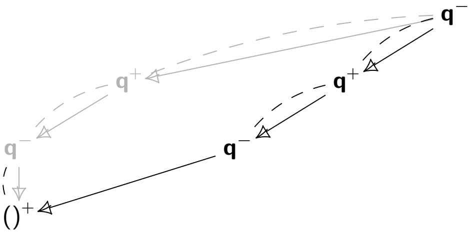

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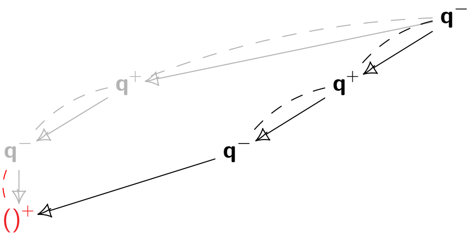

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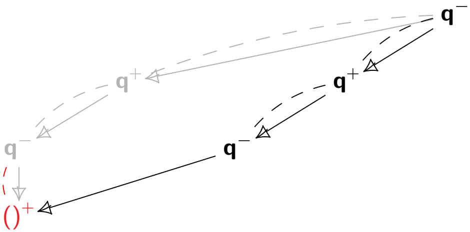

Definition

A **grounded causal chain (gcc)** of augmentation $\mathbf{q} \in \mathbf{Aug}(A)$ is

$$\rho \quad = \quad \rho_1 \rightarrow_{\mathbf{q}} \rho_2 \rightarrow_{\mathbf{q}} \ldots \rightarrow_{\mathbf{q}} \rho_n$$

where ρ_1 is minimal in $\mathbf{q}$.

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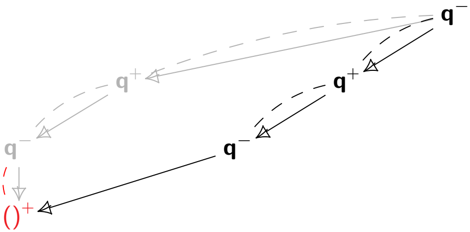

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where ρ_1 is minimal in $\mathbf{q}$.

Definition

A strategy $\sigma : A$ is **visible** iff for all $\rho \in \mathrm{gcc}(\sigma)$, $\rho \in \mathcal{C}(A)$.

Full abstraction for $\mathbf{PCF}_\parallel$ [5]

Theorem

The model $\mathbf{CG_{inn}}$ *of* **games** *and* **deterministic, (visible) parallel innocent strategies** *is intensionally fully abstract for* $\mathbf{PCF}_\parallel$.

Proof.

Via finite definability up to observational equivalence. □

[5] S. Castellan, P. C., G. Winskel. *The parallel intensionally fully abstract games model of PCF*, LICS 2015.

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\begin{array}{c}\mathrm{PCF}\\ ?\end{array} \arrow[u, shift left, "+\text{parallelism}"] \arrow[r, shift left, "+\text{state}"] & \begin{array}{c}\mathrm{IA}\\ \mathrm{CG}_{\mathrm{seq}}\end{array} \arrow[l, shift left, "+\text{parallel innocence}"] \arrow[u, shift left, "+\text{parallelism}"]
\end{tikzcd}
$$

Sequentiality and full abstraction for **IA**[6]

Theorem

The model $\mathbf{CG_{seq}}$ *of* **games** *and* **deterministic sequential strategies** *is intensionally fully abstract for* **IA**.

Proof.

If $\sigma : A$ is well-bracketed **sequential deterministic**, then

$$\mathbf{AltPlays}(\sigma) = \cup\{\mathbf{AltPlays}(\mathbf{q}) \mid \mathbf{q} \in \sigma\}$$

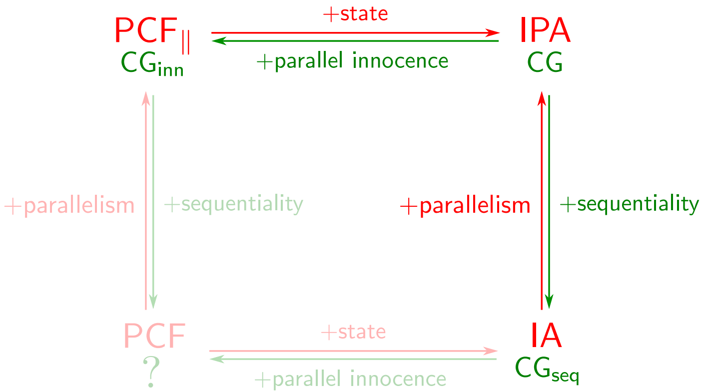
is a strategy in the sense of Abramsky-McCusker. This forms a functor

$$\mathbf{AltPlays}(-) : \mathbf{CG_{seq}} \to \mathbf{AM}$$

preserving the interpretation. $\square$

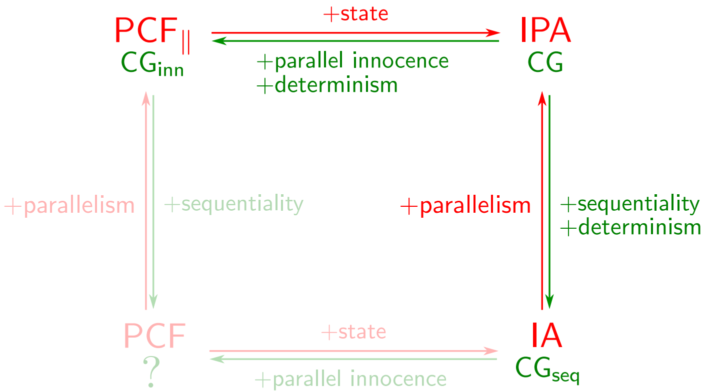
[6] S. Abramsky, G. McCusker. *Linearity, sharing and state: a fully abstract game semantics for Idealized Algol with active expressions.* 1997

Wrapping up



+state
$PCF_{\|}$
CG_{inn}
IPA
CG
+parallel innocence
+parallelism
+sequentiality
+parallelism
+sequentiality
PCF
?
+state
IA
CG_{seq}
+parallel innocence

Wrapping up

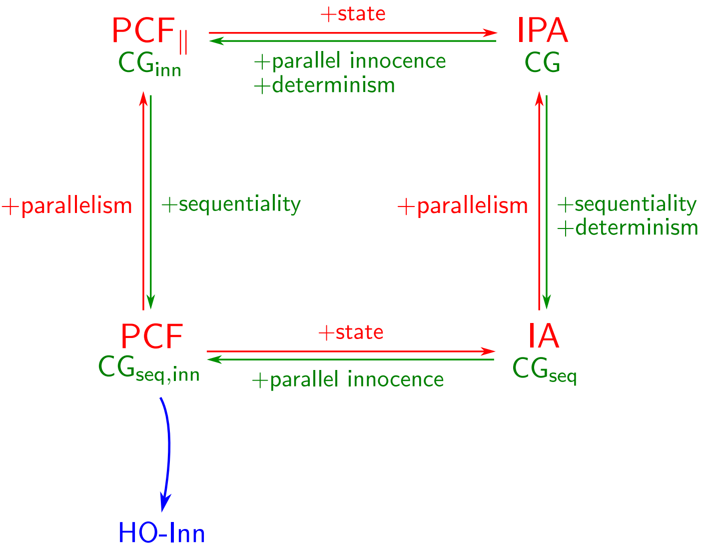


$\mathrm{PCF}_{\parallel}$
$\mathrm{CG}_{\mathrm{inn}}$
+state
+parallel innocence
+determinism
IPA
CG
+parallelism
+sequentiality
+parallelism
+sequentiality
+determinism
PCF
?
+state
+parallel innocence
IA
$\mathrm{CG}_{\mathrm{seq}}$

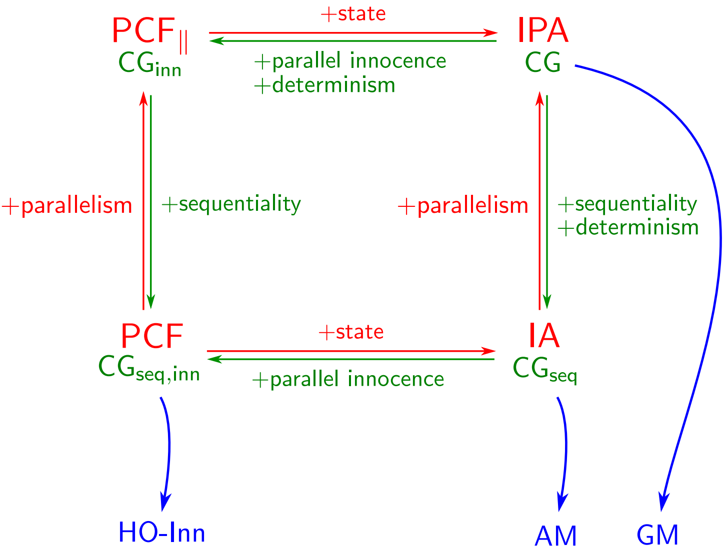
Wrapping up

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\substack{\mathsf{PCF}_{\parallel} \\ \mathsf{CG}_{\mathsf{inn}}} \arrow[r, shift left, "+\text{state}"] \arrow[d, shift left, "+\text{sequentiality}"] & \substack{\mathsf{IPA} \\ \mathsf{CG}} \arrow[l, shift left, "\substack{+\text{parallel innocence} \\ +\text{determinism}}"] \arrow[d, shift left, "\substack{+\text{sequentiality} \\ +\text{determinism}}"] \\
\substack{\mathsf{PCF} \\ \mathsf{CG}_{\mathsf{seq,inn}}} \arrow[u, shift left, "+\text{parallelism}"] \arrow[r, shift left, "+\text{state}"] & \substack{\mathsf{IA} \\ \mathsf{CG}_{\mathsf{seq}}} \arrow[l, shift left, "+\text{parallel innocence}"] \arrow[u, shift left, "+\text{parallelism}"]
\end{tikzcd}
$$

Wrapping up


$\mathsf{PCF}_{\parallel}$
$\mathsf{CG}_{\mathsf{inn}}$
+state
+parallel innocence
+determinism
IPA
CG
+parallelism
+sequentiality
+parallelism
+sequentiality
+determinism
PCF
$\mathsf{CG}_{\mathsf{seq,inn}}$
+state
+parallel innocence
IA
$\mathsf{CG}_{\mathsf{seq}}$
HO-Inn

Wrapping up



$PCF_{\parallel}$
CG_{inn}
+state
IPA
CG
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+determinism
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+sequentiality
+parallelism
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+determinism
PCF
$CG_{seq,inn}$
+state
IA
CG_{seq}
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HO-Inn
AM
GM

Wrapping up

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
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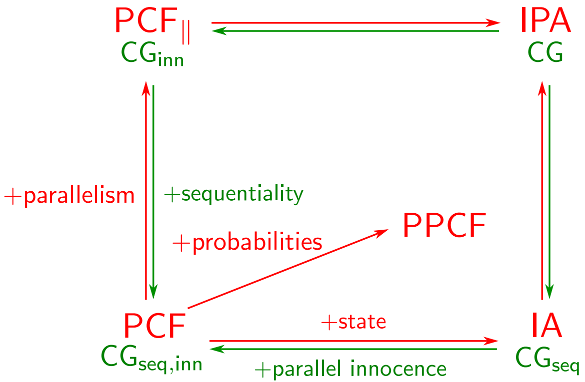
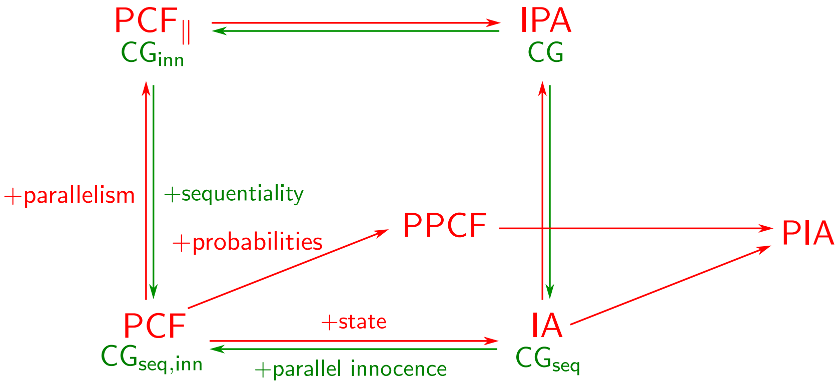
$$

Wrapping up



$\mathrm{PCF}_{\|}$
$\mathrm{CG}_{\mathrm{inn}}$
IPA
CG
+parallelism
+sequentiality
PPCF
+probabilities
PCF
$\mathrm{CG}_{\mathrm{seq,inn}}$
+state
IA
$\mathrm{CG}_{\mathrm{seq}}$
+parallel innocence

Wrapping up



$\mathsf{PCF}_{\|}$
$\mathsf{CG}_{\mathsf{inn}}$
IPA
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+parallelism
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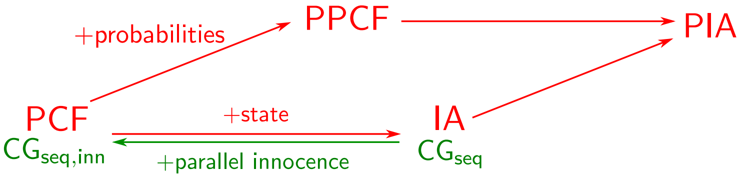
+parallel innocence

Wrapping up



PPCF
PIA
+probabilities
PCF
$CG_{seq,inn}$
+state
+parallel innocence
IA
CG_{seq}

III. The sequential face

Probabilistic **IA**

Types.

$$A, B \quad ::= \quad \mathbb{U} \mid \mathbb{B} \mid A \to B$$

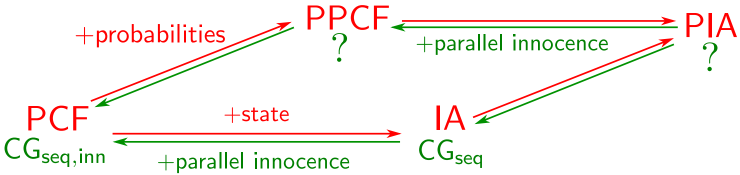
Terms.

$$
\begin{array}{lcll}
M, N & ::= & x \mid M\,N \mid \lambda x.\,M \mid Y & \lambda \\
 & & \mid \mathbf{tt} \mid \mathbf{ff} \mid \mathbf{if}\ M\ N_1\ N_2 & \\
 & & \mid \mathbf{skip} \mid M; N & \mathbf{PCF} \\
 & & \mid \mathbf{newref}\ v := b\ \mathbf{in}\ M \mid M := N \mid\ !M & +\mathbf{state} \\
 & & \mid \mathtt{rand}() & +\mathbf{probabilities}
\end{array}
$$

Roadmap

$$
\begin{tikzcd}
 & \text{PPCF} \arrow[rr] & & \text{PIA} \\
\begin{array}{c}\text{PCF}\\ \text{CG}_{\text{seq,inn}}\end{array} \arrow[ur, "+\text{probabilities}"] \arrow[rr, shift left, "+\text{state}"] & & \begin{array}{c}\text{IA}\\ \text{CG}_{\text{seq}}\end{array} \arrow[ll, shift left, "+\text{parallel innocence}"] \arrow[ur] &
\end{tikzcd}
$$

Roadmap


+probabilities
PPCF
?
PIA
?
+parallel innocence
PCF
$\mathsf{CG}_{\mathsf{seq,inn}}$
+state
IA
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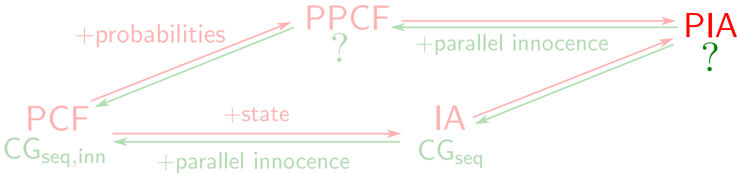
+parallel innocence

Roadmap


PPCF
?
PIA
?
+probabilities
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PCF
$CG_{seq,inn}$
+state
+parallel innocence
IA
CG_{seq}

Full abstraction for **PIA** [7]

Definition

A **probabilistic strategy** $\sigma : A$ is a function

$$\sigma : \mathbf{Aug}(A) \to [0,1]$$

satisfying some conditions.

Conjecture

The category **PCG** *of* **games** *and* **(well-bracketed) sequential probabilistic strategies** *is intensionally fully abstract for* **PIA**.

Proof.

If $\sigma : A$ is a probabilistic concurrent strategy, then setting

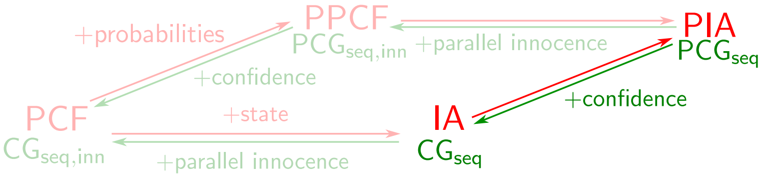
$$\begin{array}{rclcl} \mathbf{AltPlays}(\sigma) & : & \mathbf{AltPlays}(A) & \to & [0,1] \\ & & s & \mapsto & \sum_{\substack{\mathbf{q}\in\sigma \\ s\in\mathbf{AltPlays}(\mathbf{q}) \\ |s|=|\mathbf{q}|}} \sigma(\mathbf{q}) \end{array}$$

yields a probabilistic strategy in the sense of Danos-Harmer. This induces

$$\mathbf{AltPlays}(-) : \mathbf{PCG} \to \mathbf{DH}\,. \qquad \square$$

[7] V. Danos, R. Harmer. *Probabilistic game semantics*. LICS 2000.

Roadmap


PPCF
$\mathrm{PCG}_{\mathrm{seq,inn}}$
PIA
$\mathrm{PCG}_{\mathrm{seq}}$
+probabilities
+confidence
+parallel innocence
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$\mathrm{CG}_{\mathrm{seq,inn}}$
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IA
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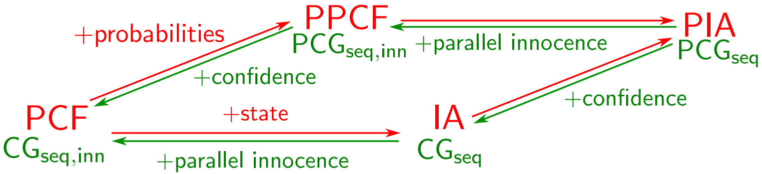
+confidence

Roadmap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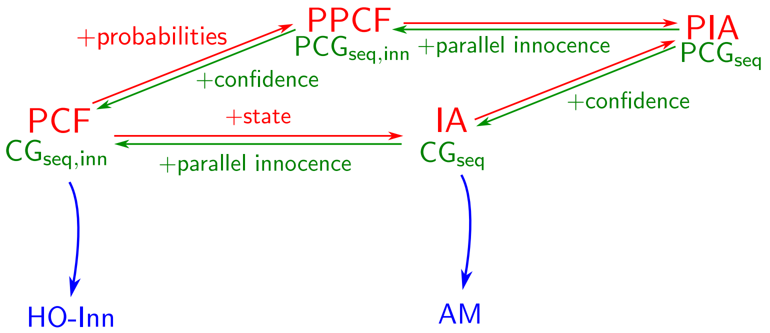
Roadmap



+probabilities
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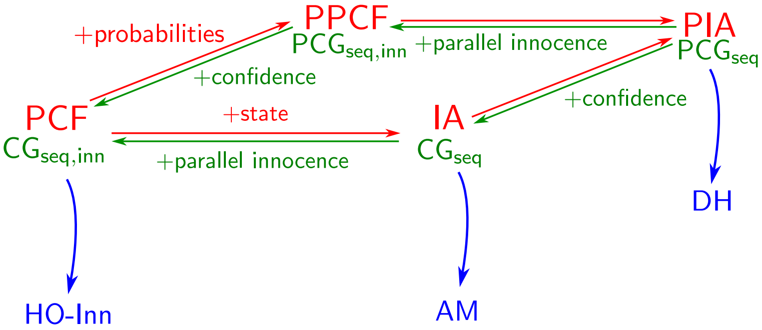
HO-Inn
AM

Roadmap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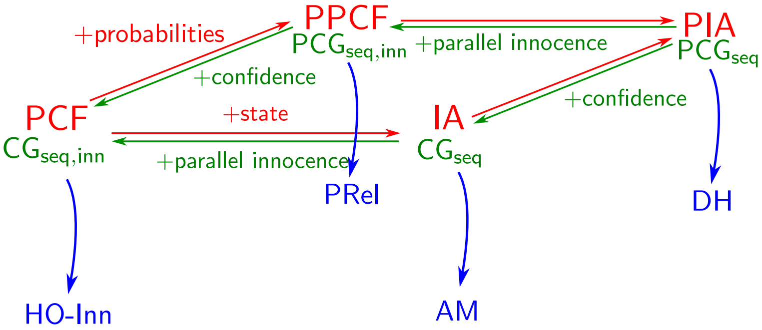
Roadmap



PPCF
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PRel
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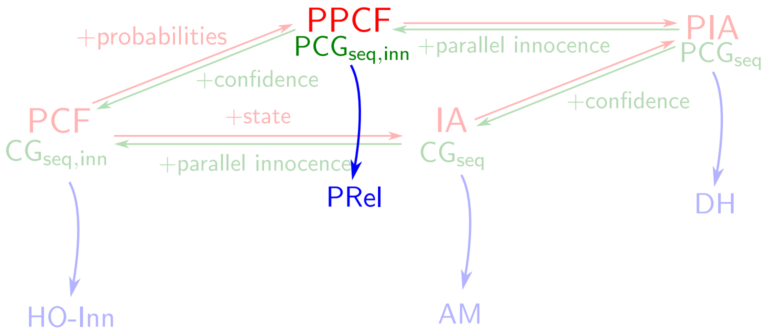
Roadmap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Refresher on the relational model

Theorem

The category **Rel** *has* **sets** *as objects and* **relations**

$$R \subseteq A \times B$$

as morphisms from A to B.
It is a **compact closed** *category with* **biproducts***.*

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$$(\!|\mathbb{B}|\!) \quad = \quad \{\mathbf{tt}, \mathbf{ff}\}$$

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$$
\begin{array}{lcl}
(\!|\mathbb{B}|\!) & = & \{\mathbf{tt}, \mathbf{ff}\} \\
(\!|\mathbb{U}|\!) & = & \{()\}
\end{array}
$$

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$$\begin{array}{rcl} (\!|\mathbb{B}|\!) & = & \{\mathbf{tt}, \mathbf{ff}\} \\ (\!|\mathbb{U}|\!) & = & \{()\} \\ (\!|A \multimap B|\!) & = & \end{array}$$

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Refresher on the relational model

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as morphisms from A *to* B.
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$$
\begin{array}{rcl}
(\!|\mathbb{B}|\!) & = & \{\mathbf{tt}, \mathbf{ff}\} \\
(\!|\mathbb{U}|\!) & = & \{()\} \\
(\!|A \multimap B|\!) & = & ((\!|A|\!) + 1) \times (\!|B|\!)
\end{array}
$$

Refresher on the relational model

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$$(\!|\lambda f^{\mathbb{B} \multimap \mathbb{U}}.\, f\ \mathbf{tt}|\!) = \left\{ \begin{array}{ccc} (\mathbb{B} & \multimap \ \mathbb{U}) \multimap & \mathbb{U} \\ ((\mathbf{tt}, & ()), & ()) \\ ((\star, & ()), & ()) \end{array} \right\}$$

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$$(\!|\lambda f^{\mathbb{B} \multimap \mathbb{U}}.\, f\ \mathbf{tt}|\!) = \left\{ \begin{array}{ccc} (\mathbb{B} & \multimap \mathbb{U}) & \multimap\ \mathbb{U} \\ ((\mathbf{tt}, & ()), & ()) \\ ((\star, & ()), & ()) \end{array} \right\}$$

$$(x, z) \in R_2 \circ R_1 \qquad \Leftrightarrow \qquad \exists y,\ (x, y) \in R_1\ \&\ (y, z) \in R_2$$

Types as games

$$[\![\mathbb{U}]\!] = \begin{tikzcd} \mathbf{q}^- \arrow[d, dash, dashed] \\ ()^+ \end{tikzcd} \qquad\qquad [\![\mathbb{B}]\!] = \begin{tikzcd} & \mathbf{q}^- \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\ \mathbf{tt}^+ \arrow[rr, dash, squiggly] & & \mathbf{ff}^+ \end{tikzcd}$$

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$$[\![\mathbb{B}]\!] = \begin{tikzcd} & \mathbf{q}^- \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\ \mathbf{tt}^+ \arrow[rr, squiggly, dash] & & \mathbf{ff}^+ \end{tikzcd}$$

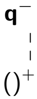
Definition

If B has exactly one minimal event;

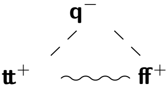
$$\begin{array}{rcl} |A \multimap B| & = & |A| + |B| \\ \mathrm{pol}_{A \multimap B} & = & [-\mathrm{pol}_A, \mathrm{pol}_B] \\ \leq_{A \multimap B} & = & \{(a_1, a_2) \mid a_1 \leq_A a_2\} \\ & \cup & \{(b_1, b_2) \mid b_1 \leq_B b_2\} \\ & \cup & \{(\min(B), a) \mid a \in |A|\} \end{array}$$

Types as games

$[\![\mathbb{U}]\!] =$



$[\![\mathbb{B}]\!] =$



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If B has exactly one minimal event;

$$
\begin{array}{rcl}
|A \multimap B| & = & |A| + |B| \\[1em]
\mathrm{pol}_{A \multimap B} & = & [-\mathrm{pol}_A, \mathrm{pol}_B] \\[1em]
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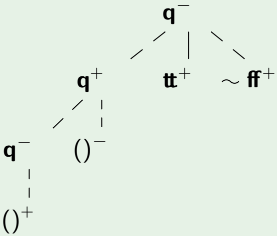
 & \cup & \{(b_1, b_2) \mid b_1 \leq_B b_2\} \\
 & \cup & \{(\min(B), a) \mid a \in |A|\}
\end{array}
$$

Example

$$[\![(\mathbb{U} \multimap \mathbb{U}) \multimap \mathbb{B}]\!] =$$



Types as games

$$[\![\mathbb{U}]\!] = \begin{tikzcd} \mathbf{q}^{-,\textcolor{red}{\mathcal{Q}}} \arrow[d, dash, dashed] \\ ()^{+} \end{tikzcd} \qquad [\![\mathbb{B}]\!] = \begin{tikzcd} & \mathbf{q}^{-} \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\ \mathtt{tt}^{+} \arrow[rr, dash, squiggly] & & \mathtt{ff}^{+} \end{tikzcd}$$

Definition

If B has exactly one minimal event;

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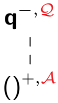
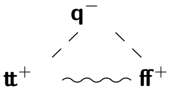
Example

$$[\![(\mathbb{U} \multimap \mathbb{U}) \multimap \mathbb{B}]\!] = $$

$$\begin{tikzcd} & & \mathbf{q}^{-} \arrow[dl, dash, dashed] \arrow[d, dash] \arrow[dr, dash, dashed] & \\ & \mathbf{q}^{+} \arrow[dl, dash, dashed] \arrow[d, dash, dashed] & \mathtt{tt}^{+} & \sim \mathtt{ff}^{+} \\ \mathbf{q}^{-} \arrow[d, dash, dashed] & ()^{-} & & \\ ()^{+} & & & \end{tikzcd}$$

Types as games

$[\![\mathbb{U}]\!] =$ 

$[\![\mathbb{B}]\!] =$ 

Definition

If B has exactly one minimal event;

$$
\begin{array}{rcl}
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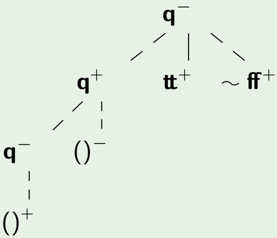
$$

Example

$[\![(\mathbb{U} \multimap \mathbb{U}) \multimap \mathbb{B}]\!] =$



Types as games

$$
[\![\mathbb{U}]\!] = \begin{tikzcd} \mathbf{q}^{-,\textcolor{red}{\mathcal{Q}}} \arrow[d, dash, dashed] \\ ()^{+,\textcolor{red}{\mathcal{A}}} \end{tikzcd}
\qquad\qquad
[\![\mathbb{B}]\!] = \begin{tikzcd} & \mathbf{q}^{-,\textcolor{red}{\mathcal{Q}}} \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\ \mathbf{tt}^{+,\textcolor{red}{\mathcal{A}}} \arrow[rr, squiggly, dash] & & \mathbf{ff}^{+,\textcolor{red}{\mathcal{A}}} \end{tikzcd}
$$

Definition

If B has exactly one minimal event;

$$
\begin{array}{rcl}
|A \multimap B| & = & |A| + |B| \\
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\end{array}
$$

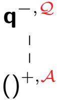
Example

$$
[\![(\mathbb{U} \multimap \mathbb{U}) \multimap \mathbb{B}]\!] =
$$

$$
\begin{tikzcd}
& & & \mathbf{q}^{-} \arrow[dl, dash, dashed] \arrow[d, dash] \arrow[dr, dash, dashed] & \\
& & \mathbf{q}^{+} \arrow[dll, dash, dashed] \arrow[d, dash, dashed] & \mathbf{tt}^{+} & \sim \mathbf{ff}^{+} \\
\mathbf{q}^{-} \arrow[d, dash, dashed] & & ()^{-} & & \\
()^{+} & & & &
\end{tikzcd}
$$

Types as games

$[\![\mathbb{U}]\!] =$



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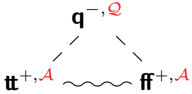

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$$
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\lambda_{A,B} &= [\lambda_A, \lambda_B]
\end{aligned}
$$

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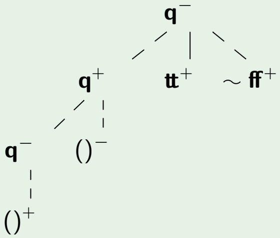

Types as games

$$[\![\mathbb{U}]\!] = \begin{tikzcd} \mathbf{q}^{-,\mathcal{Q}} \arrow[d, dash, dashed] \\ ()^{+,\mathcal{A}} \end{tikzcd} \qquad [\![\mathbb{B}]\!] = \begin{tikzcd} & \mathbf{q}^{-,\mathcal{Q}} \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\ \mathbf{tt}^{+,\mathcal{A}} \arrow[rr, no head, squiggly] & & \mathbf{ff}^{+,\mathcal{A}} \end{tikzcd}$$

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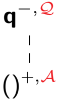
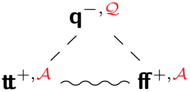
Example

$$[\![(\mathbb{U} \multimap \mathbb{U}) \multimap \mathbb{B}]\!] =$$

$$\begin{tikzcd} & & \mathbf{q}^{-,\mathcal{Q}} \arrow[dl, dash, dashed] \arrow[d, dash] \arrow[dr, dash, dashed] & \\ & \mathbf{q}^{+,\mathcal{Q}} \arrow[dl, dash, dashed] \arrow[d, dash, dashed] & \mathbf{tt}^{+,\mathcal{A}} \arrow[r, no head, squiggly] & \mathbf{ff}^{+,\mathcal{A}} \\ \mathbf{q}^{-,\mathcal{Q}} \arrow[d, dash, dashed] & ()^{-,\mathcal{A}} & & \\ ()^{+,\mathcal{A}} & & & \end{tikzcd}$$

Types as games

$[\![\mathbb{U}]\!] =$ 

$[\![\mathbb{B}]\!] =$ 

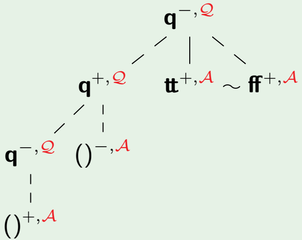
Definition

If B has exactly one minimal event;

$$
\begin{array}{rcl}
|A \multimap B| & = & |A| + |B| \\
\mathrm{pol}_{A \multimap B} & = & [-\mathrm{pol}_A, \mathrm{pol}_B] \\
\leq_{A \multimap B} & = & \{(a_1, a_2) \mid a_1 \leq_A a_2\} \\
 & \cup & \{(b_1, b_2) \mid b_1 \leq_B b_2\} \\
 & \cup & \{(\min(B), a) \mid a \in |A|\} \\
\lambda_{A,B} & = & [\lambda_A, \lambda_B]
\end{array}
$$

Example

$[\![(\mathbb{U} \multimap \mathbb{U}) \multimap \mathbb{B}]\!] =$



Definition

A configuration $x \in \mathcal{C}(A)$ is **complete** iff **every question has an answer**.
Write $\int A$ the set of non-empty complete configurations of A.

Games and the web

Theorem

For any type A,

$$\int [\![A]\!] \cong (\!|A|\!)$$

Games and the web

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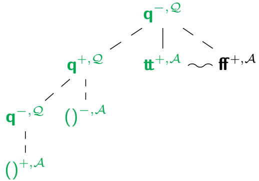

$$\sim \qquad \begin{array}{c} (\mathbb{U} \multimap \mathbb{U}) \multimap \mathbb{B} \\ (((),\ ()),\ \mathbf{tt}) \end{array}$$

Games and the web

Theorem

For any type A,

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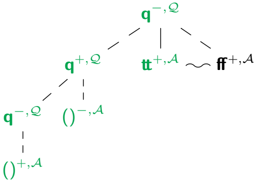
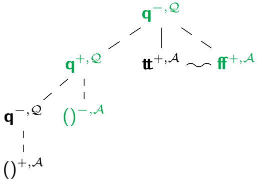


$\sim$

$$(\mathbb{U} \multimap \mathbb{U}) \multimap \mathbb{B}$$
$$(((),\ ()),\ \mathbf{tt})$$



$\sim$

$$(\mathbb{U} \multimap \mathbb{U}) \multimap \mathbb{B}$$
$$((\star,\ ()),\ \mathbf{ff})$$

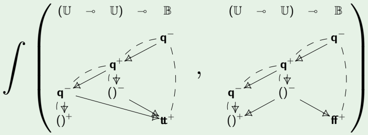
Collapse of strategies

If $\sigma : A$ is a strategy, write $\mathcal{C}(\sigma) = \cup\{\mathcal{C}(\mathbf{q}) \mid \mathbf{q} \in \sigma\}$.

Definition

$$\int \sigma = \mathcal{C}(\sigma) \cap (\int A)$$

Example



$$\sim \left\{ \begin{array}{ccccc} (\mathbb{U} & \multimap & \mathbb{U}) & \multimap & \mathbb{B} \\ (((), & & ()), & & \mathbf{tt}) \\ (((), & & ()), & & \mathbf{ff}) \end{array} \right\}$$

Composition of strategies

Definition

$\mathbf{q} \in \mathbf{Aug}(A \multimap B)$ and $\mathbf{p} \in \mathbf{Aug}(B \multimap C)$ are **causally compatible** iff

$$\begin{array}{ll} (1) & |\mathbf{q}| = x_A + x_B \ \& \ |\mathbf{p}| = x_B + x_C \\ (2) & \leq_{\mathbf{q}} \cup \leq_{\mathbf{p}} \ \text{ is } \textbf{acyclic}\,. \end{array}$$

Then, their **interaction** is

$$\mathbf{p} \circledast \mathbf{q} = (x_A + x_B + x_C, (\leq_{\mathbf{q}} \cup \leq_{\mathbf{p}})^*)$$

Their **composition** is

$$\mathbf{p} \odot \mathbf{q} = \mathbf{p} \circledast \mathbf{q} \restriction A \multimap C$$

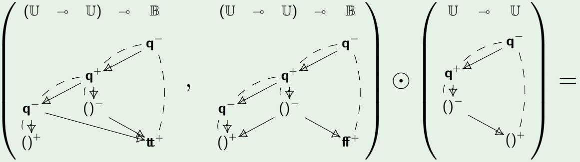
Definition

If $\sigma : A \multimap B$ and $\tau : B \multimap C$ are strategies, then their **composition** is

$$\tau \odot \sigma = \{\mathbf{p} \odot \mathbf{q} \ \mid \mathbf{q} \in \sigma \text{ and } \mathbf{p} \in \tau \text{ are } \textbf{causally compatible}\}$$

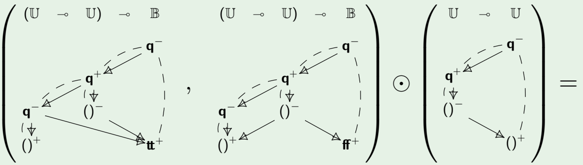
Example of composition

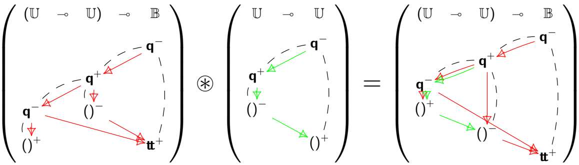
Overall composition



Example of composition

Overall composition





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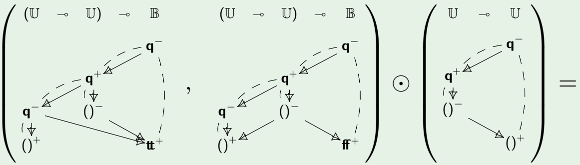
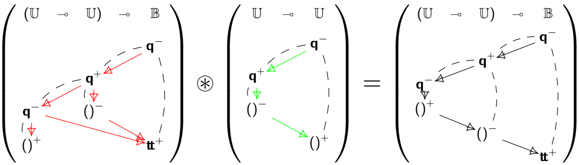
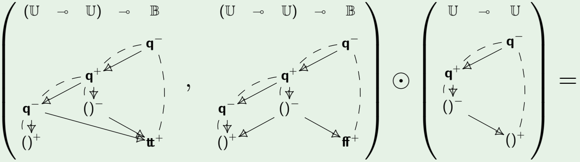
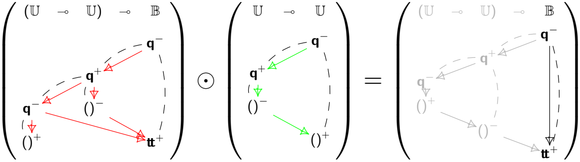




Example of composition

Overall composition





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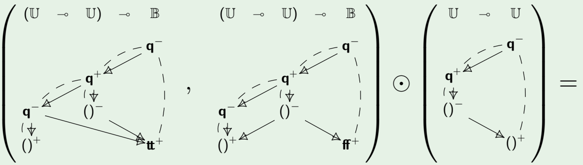
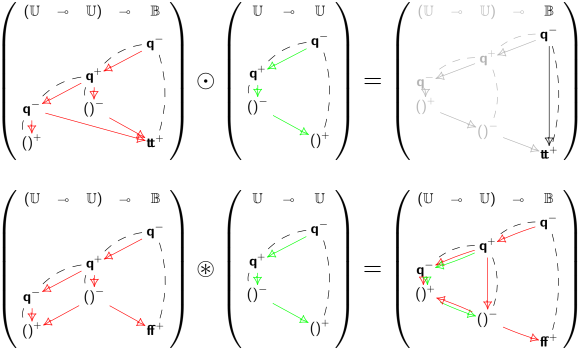
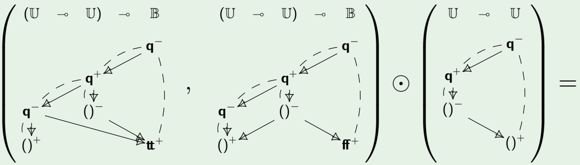
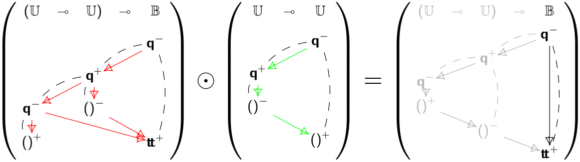


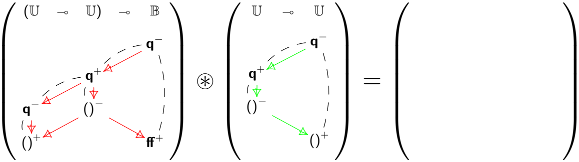


Example of composition

Overall composition







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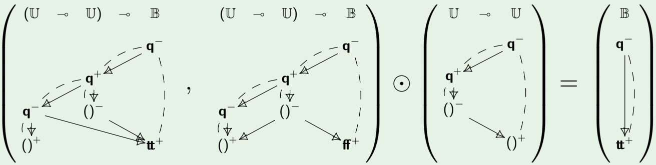

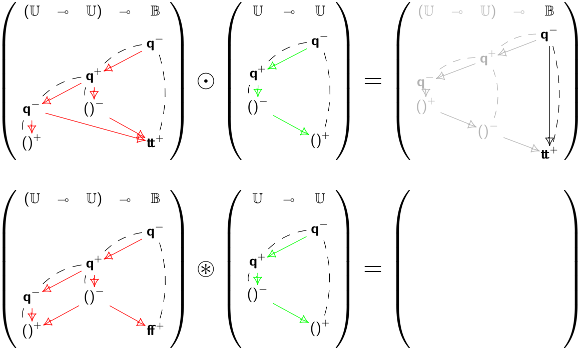

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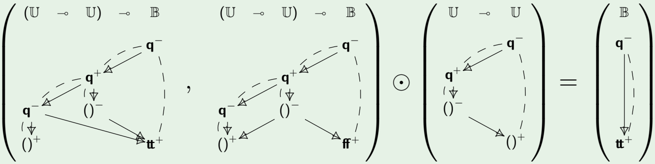
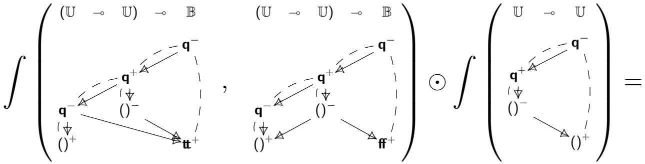
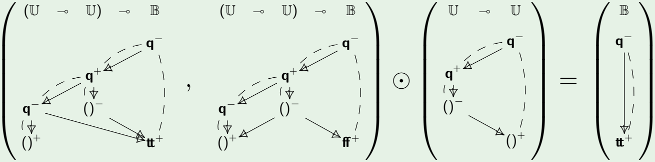
Overall composition



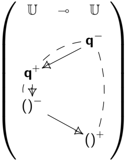


Example of composition

Overall composition



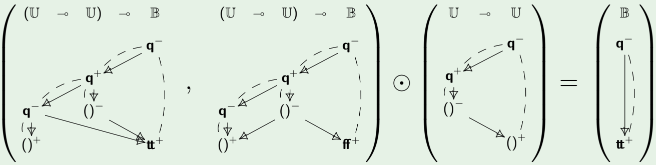
$$\left\{\begin{array}{ccccc} (\mathbb{U} & \multimap & \mathbb{U}) & \multimap & \mathbb{B} \\ (((), & & ()), & & \mathbf{tt}) \\ (((), & & ()), & & \mathbf{ff}) \end{array}\right\} \odot \int$$



$=$

Example of composition

Overall composition



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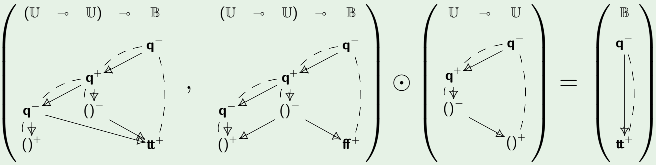
Example of composition

Overall composition



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The deadlock-free lemma [8] [9] [10]

Lemma

For $\sigma : A \multimap B$, $\tau : B \multimap C$ **visible** *strategies,* $\mathbf{q} \in \sigma$ *and* $\mathbf{p} \in \tau$ *such that*

$$(1) \quad |\mathbf{q}| = x_A + x_B \;\&\; |\mathbf{p}| = x_B + x_C \,,$$

then, $\mathbf{p}$ *and* $\mathbf{q}$ *satisfy:*

$$(2) \quad \leq_{\mathbf{q}} \cup \leq_{\mathbf{p}} \;\; \textit{is } \textbf{acyclic}$$

Proof.

By descent on the justification pointers. □

[8] P. Baillot, V. Danos, T. Ehrhard, L. Regnier, *Timeless games*, CSL 1997

[9] P.-A. Melliès, *Asynchronous games 4: A fully complete model of propositional linear logic*, LICS 2005.

[10] P. Boudes, *Thick subtrees, games and experiments*, TLCA 2009.

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Proof.

By descent on the justification pointers. $\square$

Theorem

$$\int(-) : \mathbf{CG_{vis}} \to \mathbf{Rel}$$

[8] P. Baillot, V. Danos, T. Ehrhard, L. Regnier, *Timeless games*, CSL 1997

[9] P.-A. Melliès, *Asynchronous games 4: A fully complete model of propositional linear logic*, LICS 2005.

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Theorem

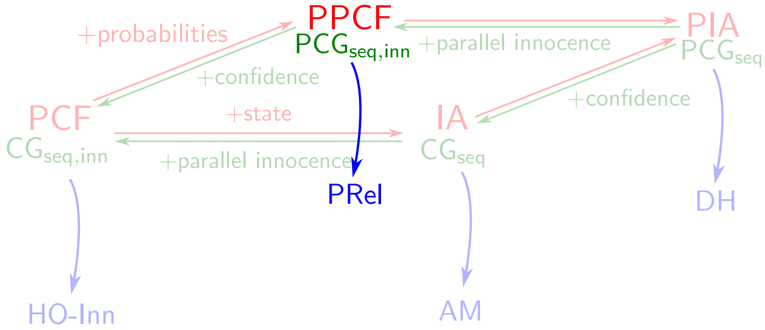
$$\textstyle\int(-) : \mathbf{CG}_{\mathbf{inn}} \to \mathbf{Rel}$$

[8] P. Baillot, V. Danos, T. Ehrhard, L. Regnier, *Timeless games*, CSL 1997

[9] P.-A. Melliès, *Asynchronous games 4: A fully complete model of propositional linear logic*, LICS 2005.

[10] P. Boudes, *Thick subtrees, games and experiments*, TLCA 2009.

Adding probabilities



The probabilistic relational model [11]

Definition

PRel has **sets** as objects, and as morphisms from A to B, **matrices**

$$(\alpha_{a,b})_{(a,b)\in A\times B} \quad \in \quad \overline{\mathbb{R}_+}^{A\times B}$$

with coefficients in $\overline{\mathbb{R}_+}$ the completed positive reals.

Definition

$$(\beta \circ \alpha)_{a,c} = \sum_{b\in B} \alpha_{a,b} \cdot \beta_{b,c}$$

Theorem (Ehrhard, Tasson, Pagani)

PRel *is fully abstract for* **PPCF***.*

[11] T. Ehrhard, C. Tasson, M. Pagani. *Probabilistic coherence spaces are fully abstract for probabilistic PCF.* POPL 2014.

Probabilistic collapse [12]

Theorem

$\mathbf{PCG}_{\mathbf{inn}}$ *is intensionally fully abstract for* **PPCF**.

Proof.

If $\sigma : A \multimap B$ and $x_A \in \int A$, $x_B \in \int B$, we define

$$(\textstyle\int \sigma)_{x_A, x_B} = \sum_{\substack{\mathbf{q} \in \sigma \\ |\mathbf{q}| = x_A + x_B}} \sigma(\mathbf{q})$$

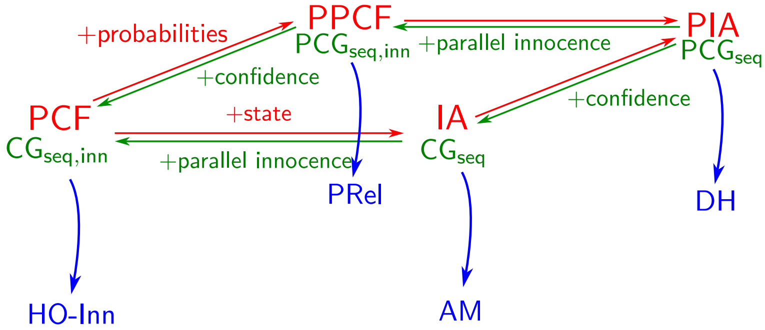
This yields a functor

$$\textstyle\int(-) : \mathbf{PCG}_{\mathbf{inn}} \to \mathbf{PRel}$$

preserving the interpretation. $\square$

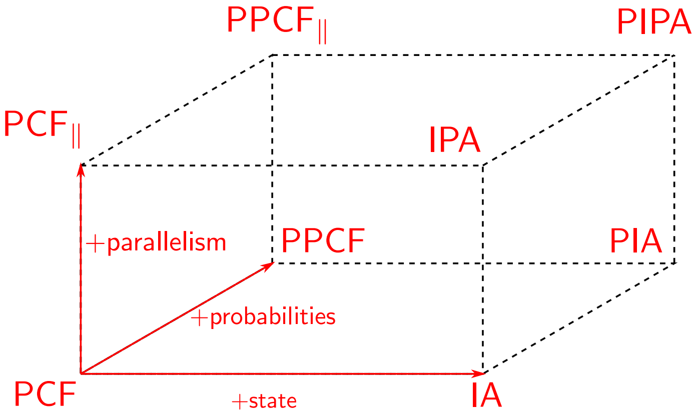
[12] S. Castellan, P. C., H. Paquet, G. Winskel. *The concurrent game semantics of Probabilistic PCF*, LICS 2018.

The sequential face



IV. Conclusions

Summary of the talk



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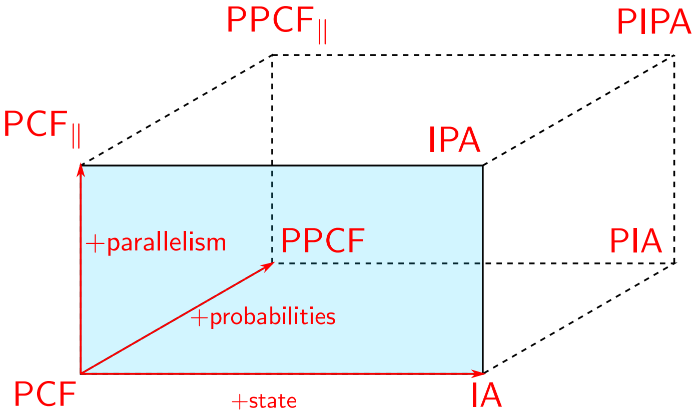

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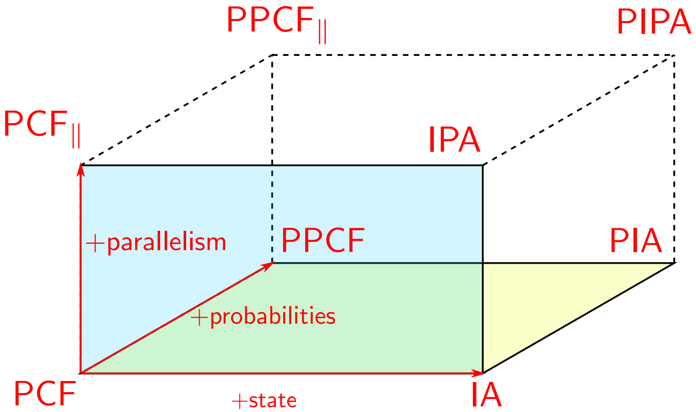

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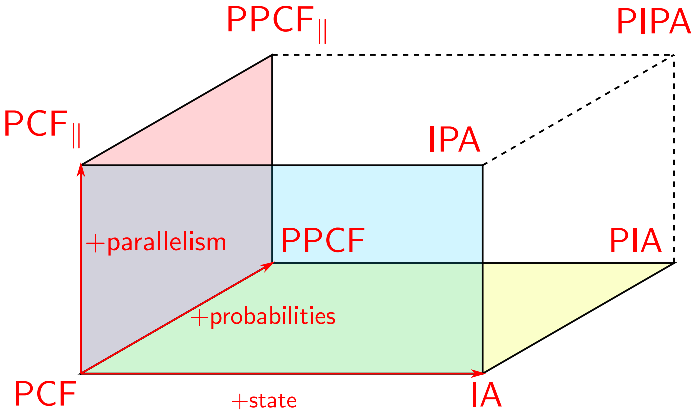
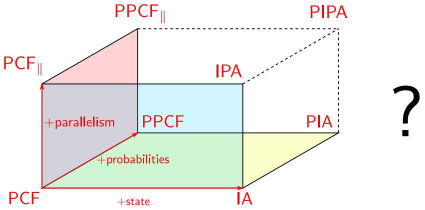
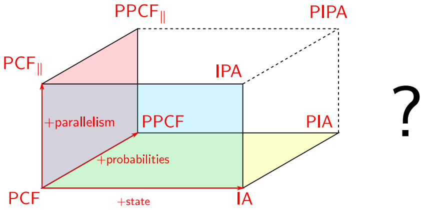


Perspectives [13]



[13] M. de Visme, *Event structures for Mixed Choice*. CONCUR 2019.

Perspectives [13]



New ANR! **DyVerSe**

[13] M. de Visme, *Event structures for Mixed Choice*. CONCUR 2019.