Loader and Urzyczyn are Logically Related

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In the beginning was untyped λ-calculus...

$$\Lambda : \quad M, N ::= x \mid \lambda x.M \mid MN$$

$$(\beta) \quad (\lambda x.M)N = M[N/x]$$

$$(\eta) \quad \lambda x.Mx = M \text{ where } x \notin fv(M)$$

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Simply Typed Lambda Calculus $\Lambda_{\rightarrow}$

Simple Types:

$$\mathbb{T}^o : \quad A, B, C ::= o \mid A \rightarrow B$$

Derivation Rules:

$$x_1 : A_1, \ldots, x_n : A_n \vdash x_i : A_i$$

$$\frac{\Delta \vdash M : A \rightarrow B \quad \Delta \vdash N : A}{\Delta \vdash MN : B} \qquad \frac{\Delta, x : A \vdash M : B}{\Delta \vdash \lambda x.M : A \rightarrow B}$$

A λ-term M is simply typable if $\exists \Delta, A$ such that $\Delta \vdash M : A$.

Basic Properties

- M is simply typable entails M is strongly normalizable (SN),
- **But** there are SN terms that are not simply typable: $\lambda x.xx$

$$\lambda x^C.x^{A \rightarrow B}x^A, \text{ one needs } C = A \text{ and } C = A \rightarrow B \ \lightning$$

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The Full Model of $\Lambda_{\rightarrow}$

The full model over a finite set X is given by

$$\mathcal{F}_X = \{\mathcal{F}_X(A)\}_{A \in \mathbb{T}^o}$$

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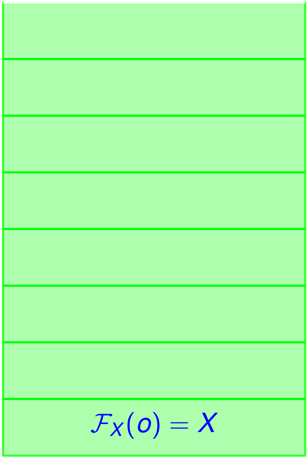

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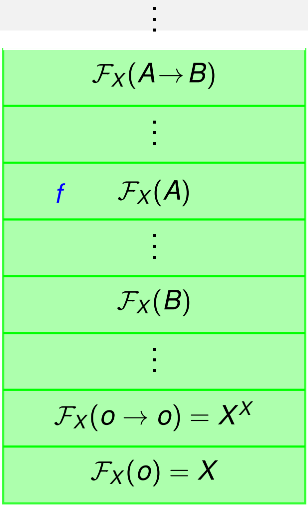

Lambda Definability

An element $f \in \mathcal{F}_X$ is λ-definable if $\exists M \in \Lambda_\to$ closed such that $[\![M]\!] = f$.

The Definability Problem…

Plotkin'73

DP: "Given an element f of any (finite) full model, is f λ-definable?"

Conjecture by Plotkin'73 and Statman'82: DP is decidable

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Theorem [Loader'01]

- Loader: DP is undecidable,

$$\begin{array}{rll} & WRP: & \text{two letters Word Rewriting Problem} \\ \leq_T & DP: & \text{Definability Problem} \end{array}$$

The Definability Problem...

Plotkin'73

DP: "Given an element f of any (finite) full model, is f λ-definable?"

Let $\mathcal{F}_n$ be the full model over $X = \{x_1, \dots, x_n\}$ for some $n > 0$.

DP$_n$: "Given an element $f \in \mathcal{F}_n$, is f λ-definable?"

Conjecture by Plotkin'73 and Statman'82: DP is decidable
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Theorem [Loader'01,Joly'03]

- Loader: DP is undecidable,
- Loader: DP$_n$ is undecidable for every $n > 6$,
- Joly: DP$_n$ is undecidable for every $n > 1$.

	WRP	:	two letters Word Rewriting Problem
$\leq_T$	*DP*	:	Definability Problem

Intersection Type Disciplines

More permissive type systems have been proposed...

CDV: Coppo, Dezani, Venneri'81

Logical characterization of Strong Normalization

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CDV: Intersection Type System

Types:

$$\mathbb{A}: \quad \alpha, \beta, \ldots \qquad \text{countable set of atoms}$$
$$\mathbb{T}_\wedge: \quad \sigma, \tau ::= \alpha \mid \sigma \to \tau \mid \sigma \wedge \tau \qquad \text{intersection types}$$

Derivation Rules:

$$x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash_\wedge x_i : \sigma_i \quad (ax)$$

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Subtyping:

$$\sigma \leq \sigma \ (\text{refl}) \qquad \sigma \wedge \tau \leq \sigma \ (\text{incl}_L) \qquad \sigma \wedge \tau \leq \tau \ (\text{incl}_R)$$

$$(\sigma \to \tau) \wedge (\sigma \to \tau') \leq \sigma \to (\tau \wedge \tau') \quad (\to_\wedge)$$

$$\frac{\sigma \leq \gamma \quad \gamma \leq \tau}{\sigma \leq \tau} (\text{trans}) \qquad \frac{\sigma \leq \tau \quad \sigma \leq \tau'}{\sigma \leq \tau \wedge \tau'} (\text{glb}) \qquad \frac{\sigma' \leq \sigma \quad \tau \leq \tau'}{\sigma \to \tau \leq \sigma' \to \tau'} (\to)$$

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CDV: Intersection Type System

A λ-term M is typable in CDV if $\exists\Gamma, \sigma$ such that $\Gamma \vdash_\wedge M : \sigma$.

Properties

- M is typable in CDV $\iff$ M is strongly normalizable

$$\lambda x^{\alpha\wedge(\alpha\to\beta)}.x^{\alpha\to\beta}x^{\alpha}$$

Type Inhabitation

An intersection type σ is inhabited if $\exists M$ closed such that $\vdash_\wedge M : \sigma$.

IHP: Inhabitation Problem

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Urzyczyn's Proof

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Refinement of IHP:

IHP$_n$: "Given an intersection type σ with at most n atoms, is σ inhabited?"

IHP: Inhabitation Problem for Game Types

Game Types: $\mathcal{G} = \mathbb{A} \cup \mathcal{B} \cup \mathcal{C}$

$$\begin{array}{rcl} \mathcal{A} & = & \mathbb{A}^\wedge \\ \mathcal{B} & = & (\mathcal{A} \to \mathcal{A})^\wedge \\ \mathcal{C} & = & (\mathcal{D} \to \mathcal{A})^\wedge \text{ for } \mathcal{D} = (\mathcal{B} \to \mathbb{A}) \wedge (\mathcal{B} \to \mathbb{A}) \end{array}$$

where $X^\wedge = \{\sigma_1 \wedge \cdots \wedge \sigma_k : k > 0,\ \sigma_i \in X\}$ and
$(X \to Y) = \{\sigma \to \tau : \sigma \in X,\ \tau \in Y\}$

Theorem [Urzyczyn'99]

- Urzyczyn: IHP is undecidable.
- Urzyczyn: IHP is already undecidable for game types.

DP $\Lambda_{\to}$ vs IHP $\Lambda_{\wedge}$

- Apparently, two unrelated problems:

Definability Problem	Inhabitation Problem
• Simply typed λ-calculus • Denotational models • Definability	• system CDV • Intersection types • Inhabitation

- Salvati's "external" viewpoint brought an unexpected link:

$$\text{DP} \simeq_T \text{IHP}$$

Ingredients:

- Uniform Intersection Types
- Monotone Finite Models
- Logical Relations

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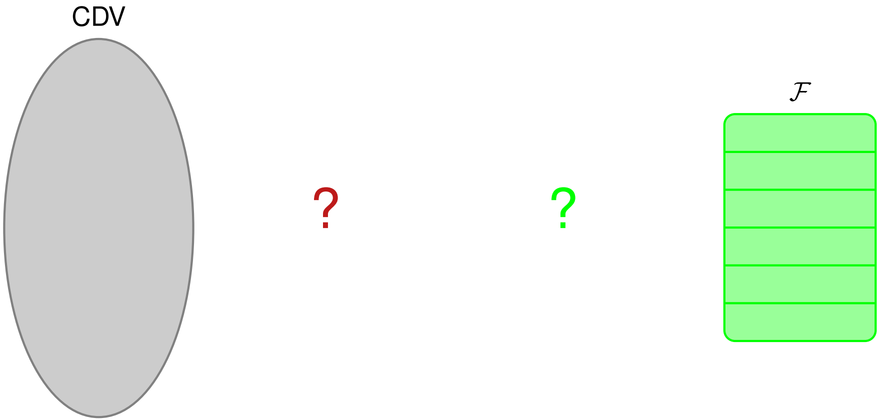

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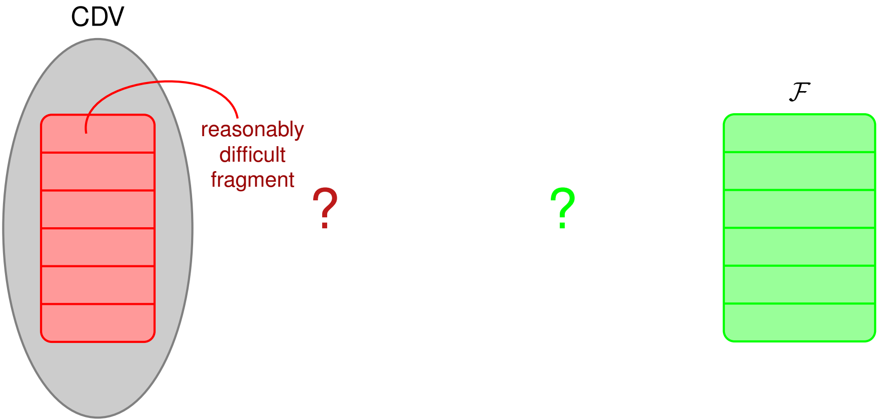

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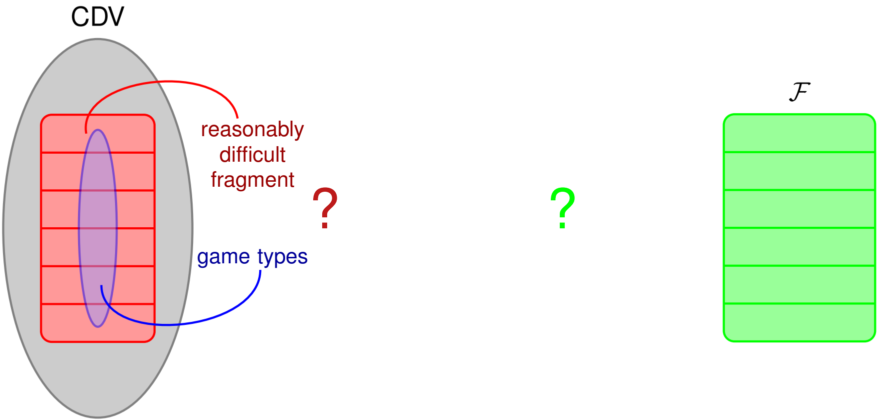

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Some intersection types

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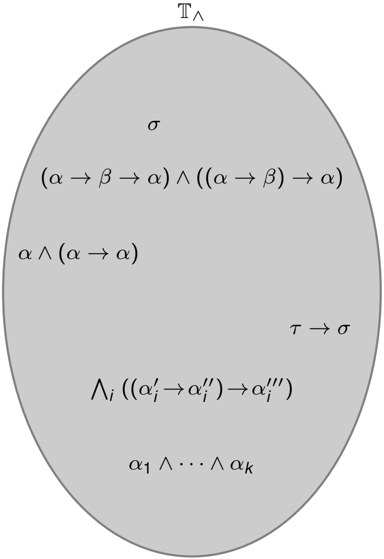
“follow the structure” of simple types.



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Intersection Types Uniform with A

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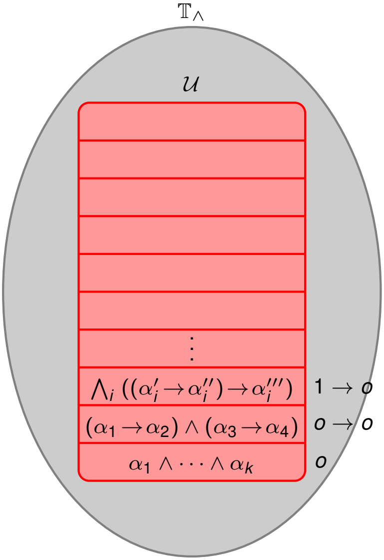

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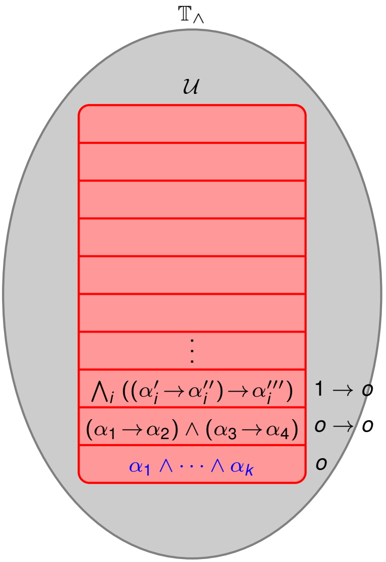

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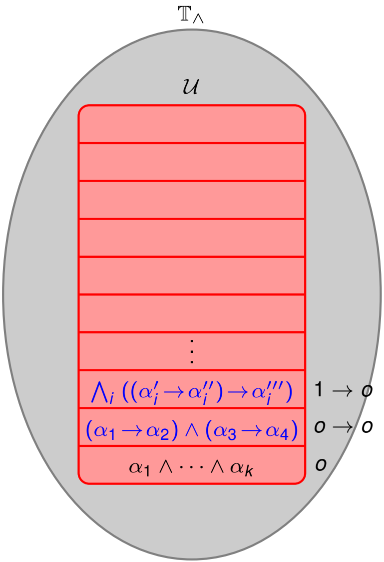

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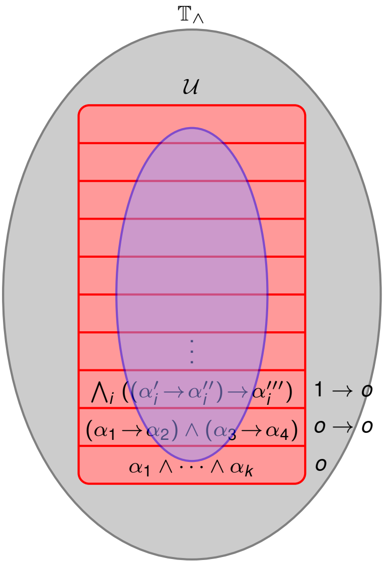
Remark Game types are uniform:

- $\mathcal{A} = \mathbb{A}^\wedge \subseteq \mathcal{U}(o)$
- $\mathcal{B} = (\mathcal{A} \to \mathcal{A})^\wedge \subseteq \mathcal{U}(o \to o)$,
- $\mathcal{C} = (\mathcal{D} \to \mathcal{A})^\wedge$
  $\subseteq \mathcal{U}(((o \to o) \to o) \to o)$



Uniform Intersection Types

Some intersection types

$$\mathbb{T}_\wedge : \quad \sigma, \tau ::= \alpha \mid \sigma \to \tau \mid \sigma \wedge \tau$$

"follow the structure" of simple types.

Intersection Types Uniform with A

- $\mathcal{U}(o) = \mathbb{A}^\wedge$
- $\mathcal{U}(B \to C) = (\mathcal{U}(B) \to \mathcal{U}(C))^\wedge$

Remark Game types are uniform:

- $\mathcal{A} = \mathbb{A}^\wedge \subseteq \mathcal{U}(o)$
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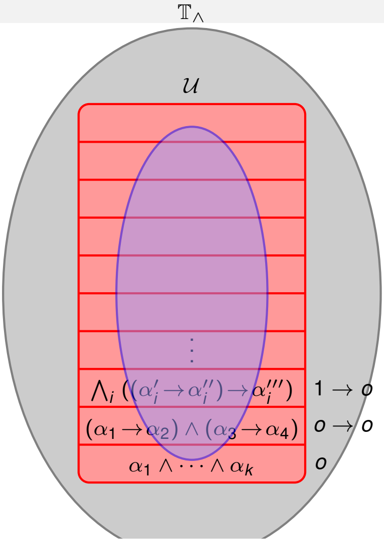

Corollary

IHP is undecidable also for Uniform Intersection Types.

CDV$^\omega$: Adding Tops to CDV

CDV$^\omega$

$$\mathbb{T}^\omega_\wedge : \quad \sigma, \tau ::= \alpha \mid \sigma \to \tau \mid \sigma \wedge \tau \mid \omega$$

Intersection Types with ω Uniform with A:

- $\mathcal{U}^\omega(o) = (\mathbb{A} \cup \{\omega\})^\wedge$
- $\mathcal{U}^\omega(B \to C) = (\mathcal{U}^\omega(B) \to \mathcal{U}^\omega(C))^\wedge$

Define

$$\omega_o = \omega \qquad \omega_{A \to B} = \omega_A \to \omega_B$$

We add to the subtyping relation of CDV:

$$\frac{\sigma \in \mathcal{U}^\omega(A)}{\sigma \leq \omega_A}\ (\leq_A)$$

Type judgments of CDV$^\omega$: $\quad \Gamma \vdash^\omega_\wedge M : \sigma$.

CDV$^\omega$ is NOT the usual Intersection Type System with ω

CDV$^\omega$: Adding Tops to CDV

CDV$^\omega$

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CDV and CDV$^\omega$ are NOT restricted to uniform types

Properties of Uniform Intersection Types

1. Let $\sigma \in \mathcal{U}^\omega(A)$ and $\tau \in \mathcal{U}^\omega(A')$, then:

$$\sigma \leq \tau \quad \Rightarrow \quad A = A'.$$

2. Given $\sigma \in \mathcal{U}^\omega(A)$ and M normal and closed:

$$\vdash^\omega_\wedge M : \sigma \quad \Rightarrow \quad \vdash M : A$$

3. Given ω-free $\sigma \in \mathcal{U}(A)$ and M normal and closed:

$$\vdash_\wedge M : \sigma \iff \vdash^\omega_\wedge M : \sigma$$

Remark: Only true for normal terms

$$\frac{\vdash^\omega_\wedge (\lambda xy.y) : \gamma \to \alpha \to \alpha \quad \vdash^\omega_\wedge (\lambda z.zz) : \gamma}{\vdash^\omega_\wedge (\lambda xy.y)(\lambda z.zz) : \alpha \to \alpha}$$

where $\gamma = (\beta \wedge (\beta \to \beta)) \to \beta$.

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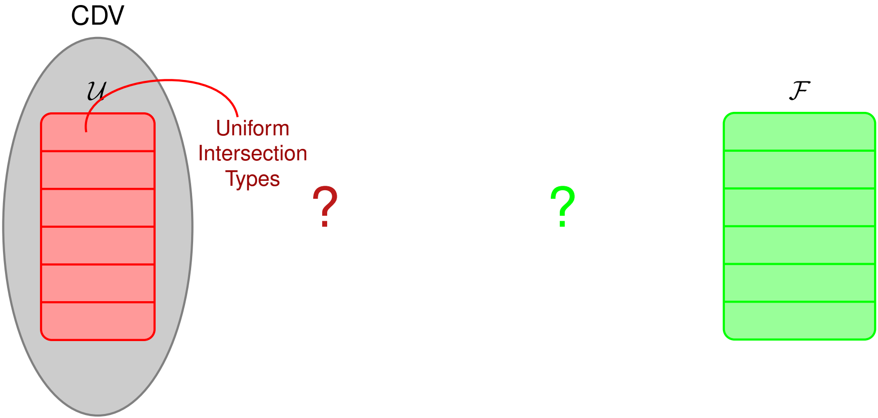
Inhabitation Reduces to Definability



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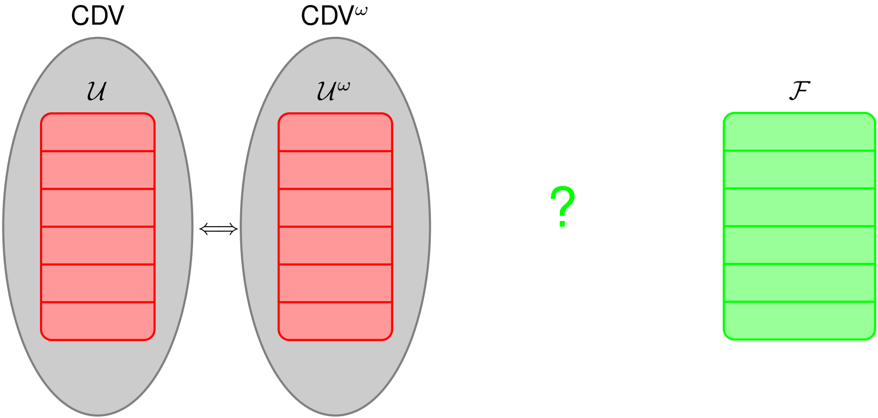

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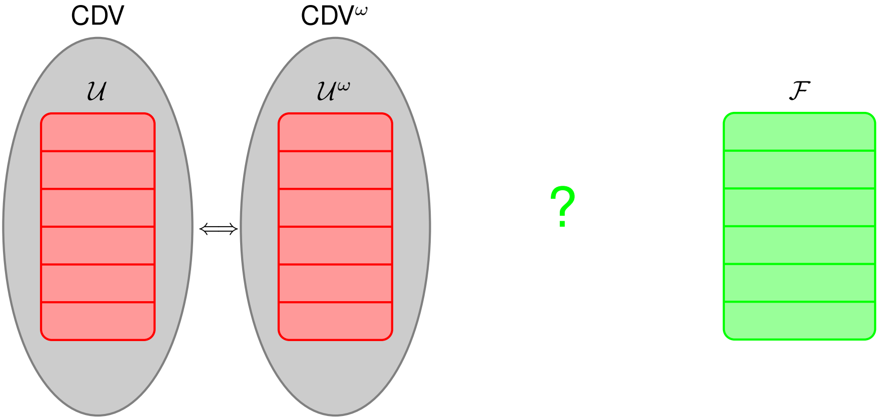
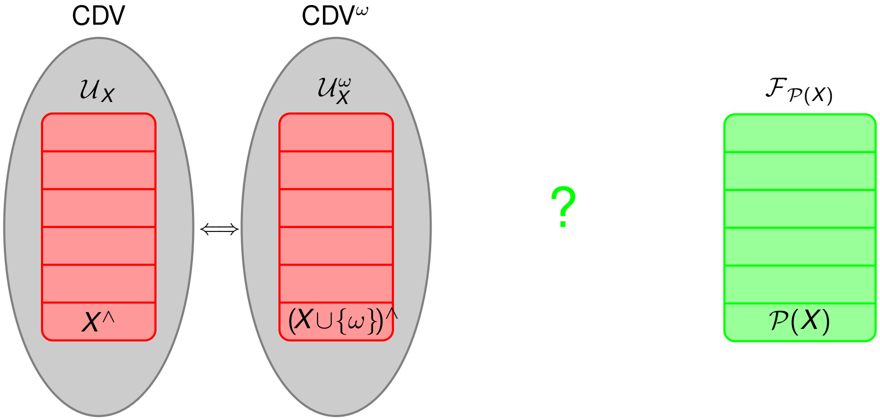


Problem: the model $\mathcal{F}$ is finite

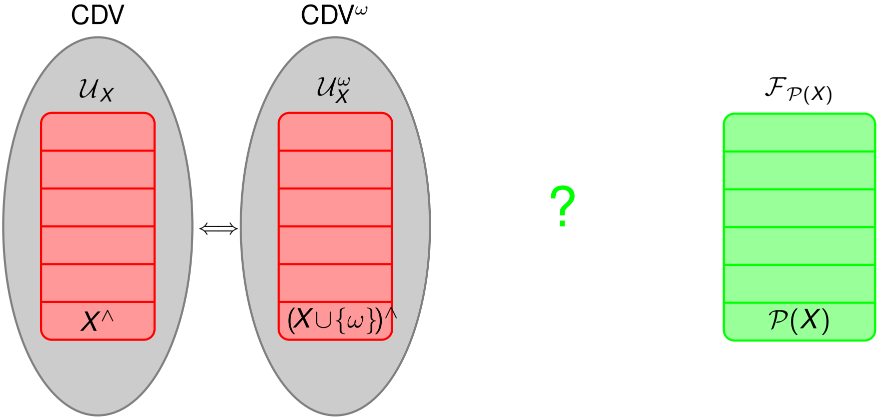
Inhabitation Reduces to Definability



Problem: the model $\mathcal{F}$ is finite

Let us consider a finite set $X \subseteq \mathbb{A}$.

Inhabitation Reduces to Definability



Problem: the model $\mathcal{F}$ is finite and very hard to study!

Let us consider a "simpler" model

The Monotone Model of $\Lambda_{\to}$

The monotone model over $(\mathcal{P}(X), \subseteq)$ is

$$\mathcal{D} = \{(\mathcal{D}_A, \sqsubseteq_A)\}_{A \in \mathbb{T}^o}$$

where

- $\mathcal{D}(o) = \mathcal{P}(X)$ and $f \sqsubseteq_o g$ iff $f \subseteq g$,

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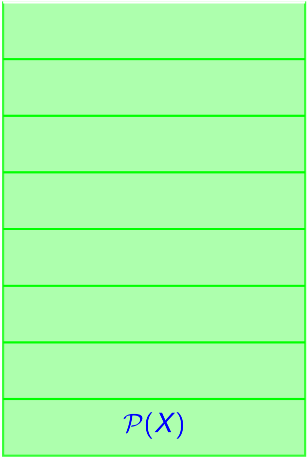

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- $\mathcal{D}_{B \to C} = [\mathcal{D}_B \to_m \mathcal{D}_C]$
  $\sqsubseteq_{B \to C} =$ pointwise ordering.

$\vdots$

$[\mathcal{D}_B \to_m \mathcal{D}_C]$
$\vdots$
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Step Functions

Let $f \in \mathcal{D}_A, g \in \mathcal{D}_B$, define $(f \mapsto g) \in \mathcal{D}_{A\to B}$:

$$(f \mapsto g)(h) = \begin{cases} g & \text{if } f \sqsubseteq_A h, \\ \bot_B & \text{otherwise.} \end{cases}$$

$\vdots$

$[\mathcal{D}_B \to_m \mathcal{D}_C]$

$\vdots$

$\mathcal{D}_C$

$\vdots$

$\mathcal{D}_B$

$\vdots$

$\mathcal{P}(X) \to_m \mathcal{P}(X)$

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Step functions are generators

For every $f \in \mathcal{D}_{A \to B}$ we have $f = \sqcup_{g \in \mathcal{D}_A}(g \mapsto f(g))$.

Intersection Types Uniform with A capture $\mathcal{D}_A$

$$(\cdot)^\bullet : \mathcal{U}^\omega(A) \to \mathcal{D}_A$$

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$\mathcal{U}^\omega(o)$

- $\mathcal{U}^\omega(o) = X \cup \{\omega\}$

$\mathcal{D}_o$

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Intersection Types Uniform with A capture $\mathcal{D}_A$

$$(\cdot)^\bullet : \mathcal{U}^\omega(A) \to \mathcal{D}_A$$

$A = 0$ $\qquad \alpha^\bullet = \{\alpha\} \qquad (\sigma \wedge \tau)^\bullet = \sigma^\bullet \cup \tau^\bullet$

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- $\alpha_1 \wedge \cdots \wedge \alpha_k$
- $\alpha \wedge \beta \leq \alpha$
- ω top

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Proposition

- For all A, $\sigma, \tau \in \mathcal{U}^\omega(A)$, we have $\sigma \leq \tau \iff \tau^\bullet \sqsubseteq \sigma^\bullet$.
- The map $(\cdot)^\bullet$ is an order-reversing bijection $(\mathcal{U}^\omega(A)/\simeq) \;\cong\; \mathcal{D}_A$

Intersection Types Uniform with A capture $\mathcal{D}_A$

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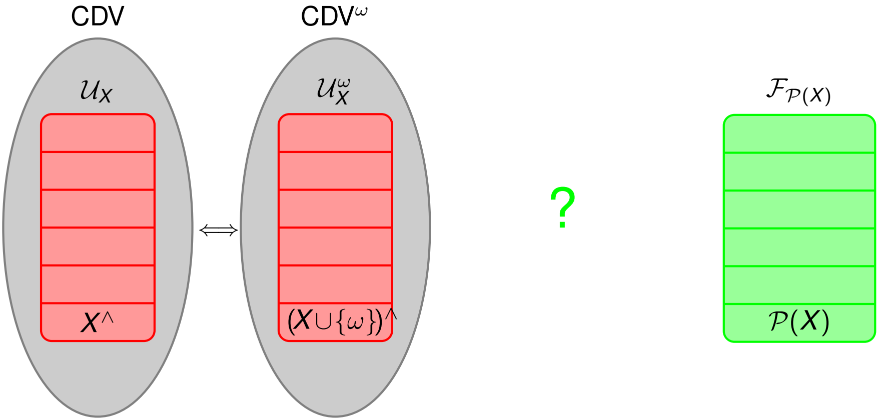
Theorem

Let M be normal, closed and such that $\vdash M : A$. Then for all $\sigma \in \mathcal{U}^\omega(A)$:

$$\vdash^\omega_\wedge M : \sigma \iff \sigma^\bullet \sqsubseteq [\![M]\!]^{\mathcal{D}}$$

Inhabitation Reduces to Definability



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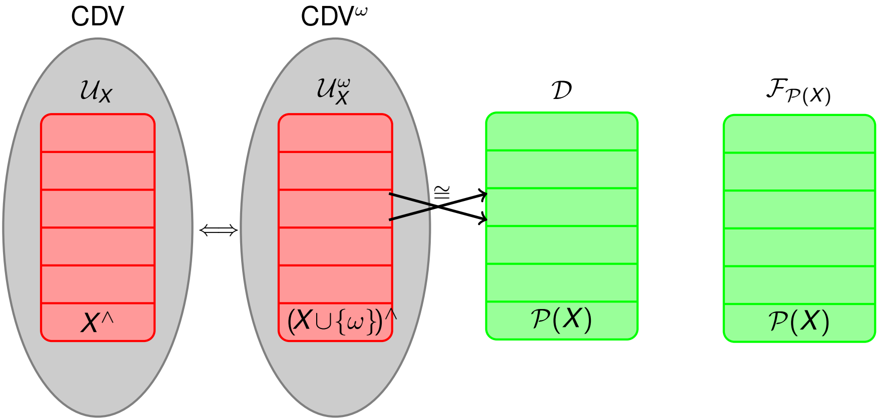

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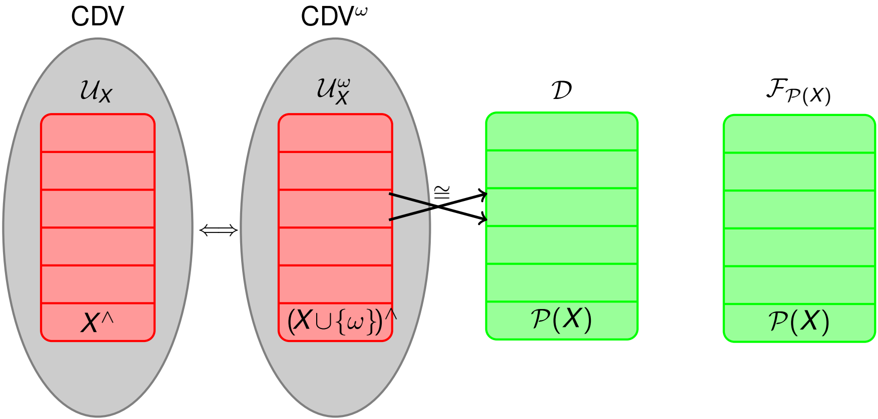

We need a link between $\mathcal{D}$ and $\mathcal{F}_{\mathcal{P}(X)}$:

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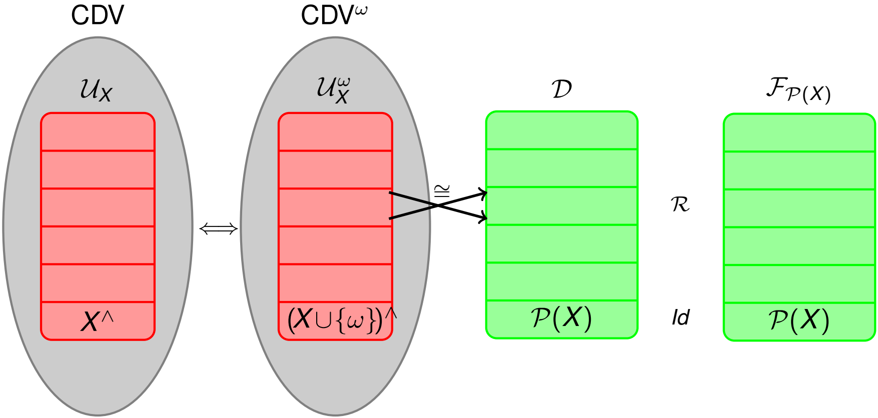
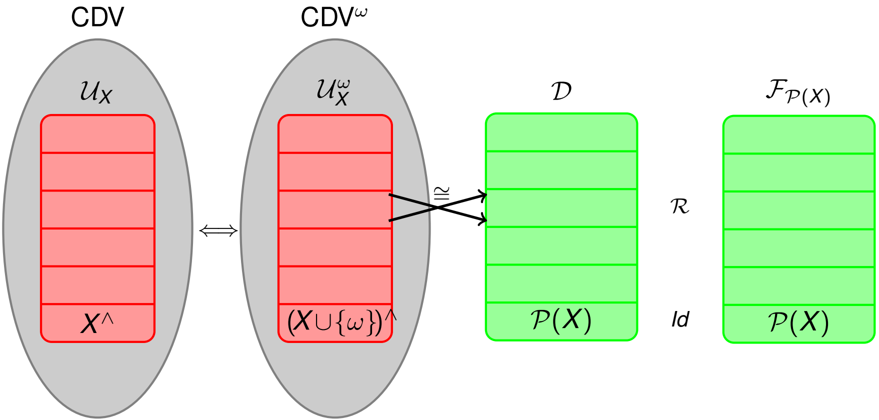


We need a link between $\mathcal{D}$ and $\mathcal{F}_{\mathcal{P}(X)}$: Logical Relations!

- $\mathcal{R}_o = Id$,
- $f\ \mathcal{R}_{A\to B}\ g \iff \forall h \in \mathcal{D}_A, h' \in \mathcal{F}_A\ [h\ \mathcal{R}_A\ h' \Rightarrow f(h)\ \mathcal{R}_B\ g(h')]$.

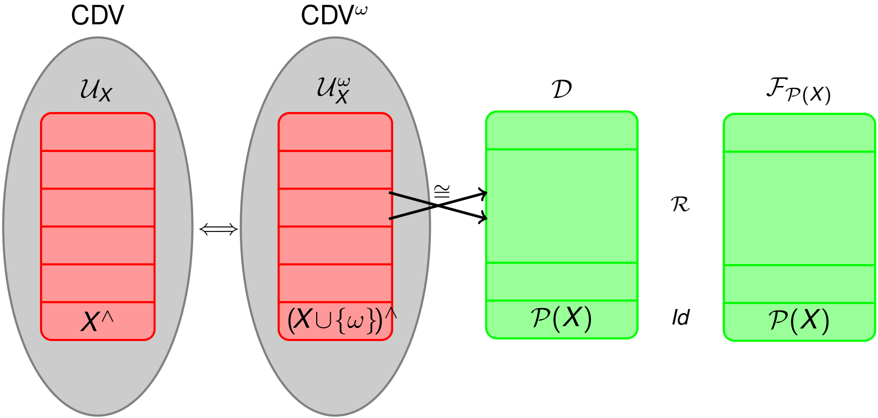
Inhabitation Reduces to Definability



We need a link between $\mathcal{D}$ and $\mathcal{F}_{\mathcal{P}(X)}$: Logical Relations!

- Fundamental Lemma: For all $M \in \Lambda_\to$ closed we have $[\![M]\!]^\mathcal{D} \; \mathcal{R} \; [\![M]\!]^\mathcal{F}$

Ready to go: $\vdash_\wedge \, ? : \sigma \in \mathcal{U}$



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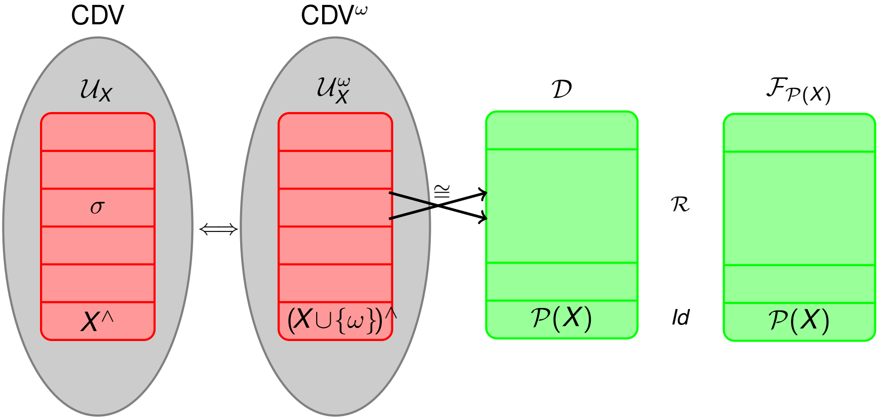

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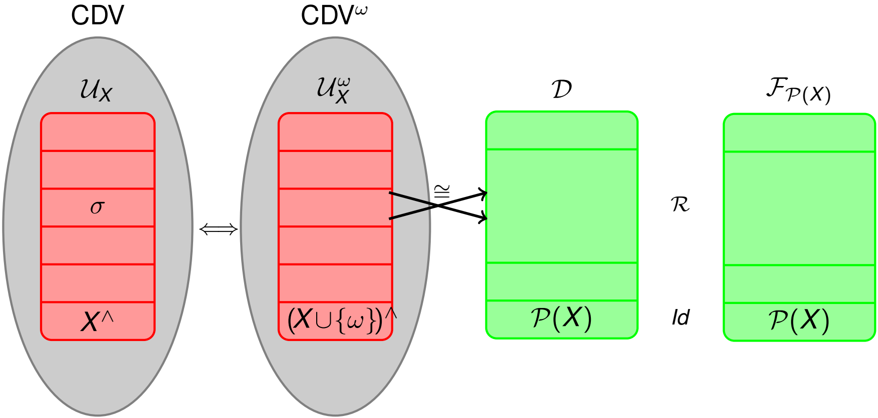
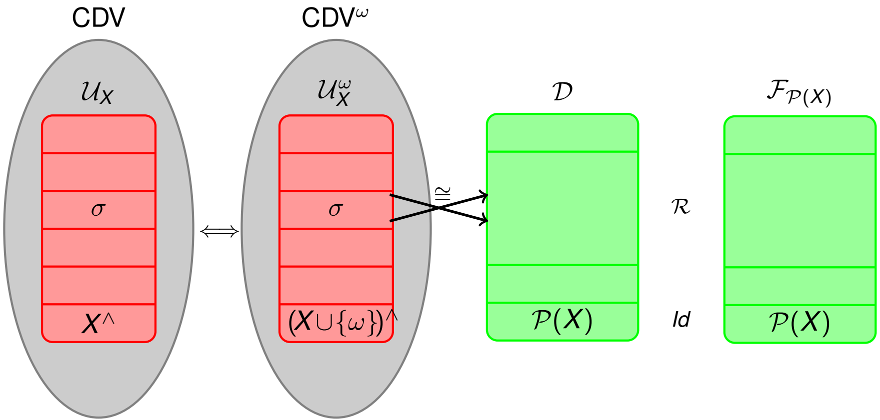


$\vdash_\wedge M : \sigma$

Focus on M simply typable and normal.

Ready to go: $\vdash_\wedge \, ? : \sigma \in \mathcal{U}$



$$\vdash_\wedge M : \sigma \quad \Longleftrightarrow \quad \vdash_\wedge^\omega M : \sigma$$

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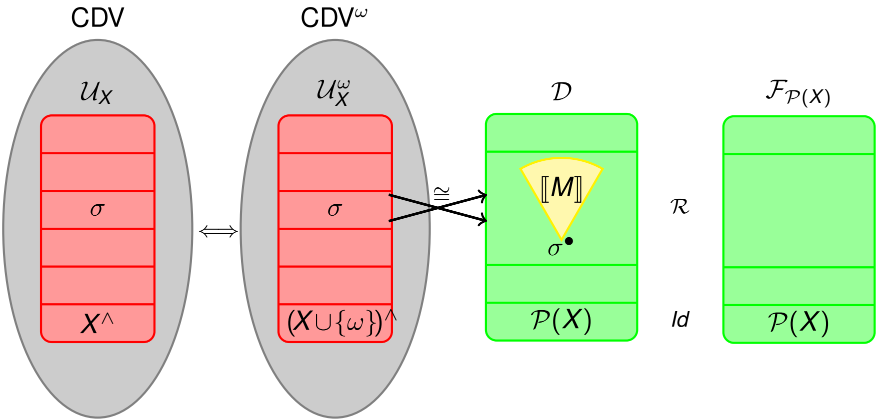
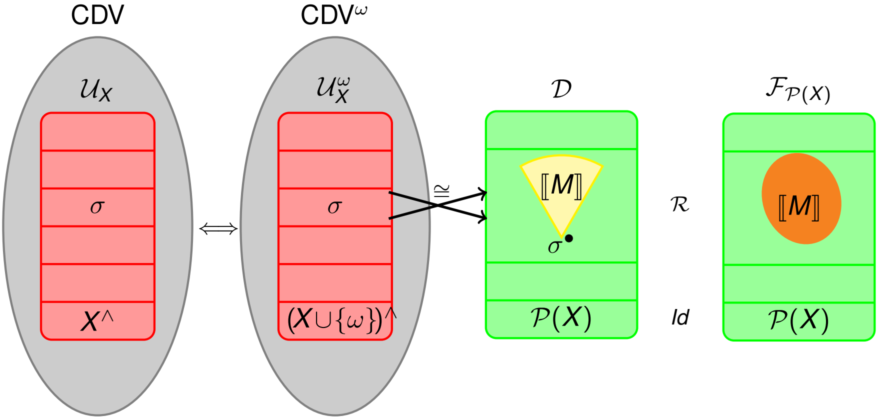


$$\vdash_\wedge M : \sigma \quad \Longleftrightarrow \quad \vdash^\omega_\wedge M : \sigma \quad \Longleftrightarrow \quad [\![M]\!]^{\mathcal{D}} \in \sigma^\bullet\!\uparrow$$

Focus on M simply typable and normal.

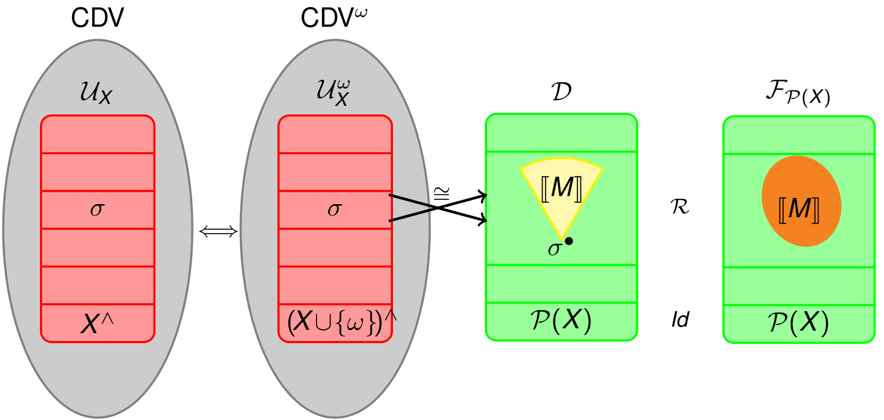
Ready to go: $\vdash_\wedge ? : \sigma \in \mathcal{U}$



$$\vdash_\wedge M : \sigma \quad \iff \quad \vdash_\wedge^\omega M : \sigma \quad \iff \quad [\![M]\!]^{\mathcal{D}} \in \sigma^\bullet\!\uparrow \quad \iff \quad [\![M]\!]^{\mathcal{F}} \in \mathcal{R}(\sigma^\bullet\!\uparrow)$$

If λ-definability is decidable, then IHP for (Uniform) Intersection Types is decidable$\lightning$ by Urzyczyn

Ready to go: $\vdash_\wedge \, ? : \sigma \in \mathcal{U}$



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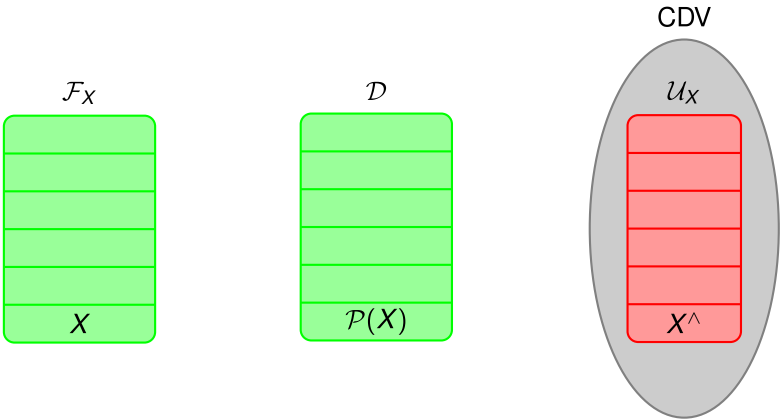
Inhabitation Problem for CDV $\leq_T$ Definability Problem

Definability Reduces to Inhabitation



Remark

- $\mathcal{F}_X$ is over X
- $\mathcal{D}$ is over $\mathcal{P}(X)$

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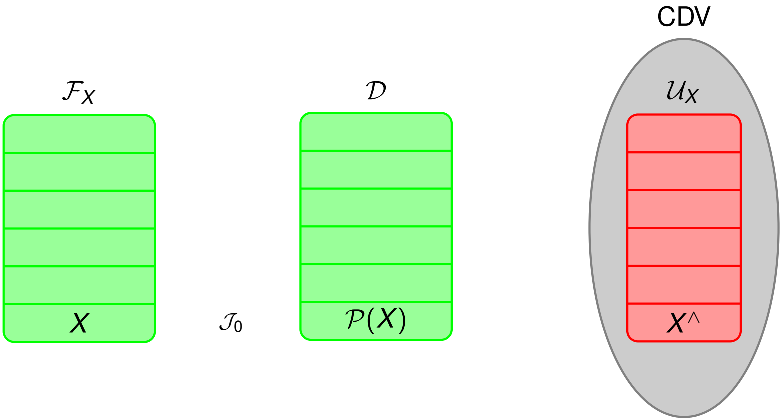

Logical Relations

- $\mathcal{J}_o = \{(f, F) \mid f \in F\}$, auxiliary relation $\mathcal{I}_o = \{(f, \{f\}) \mid f \in X\}$

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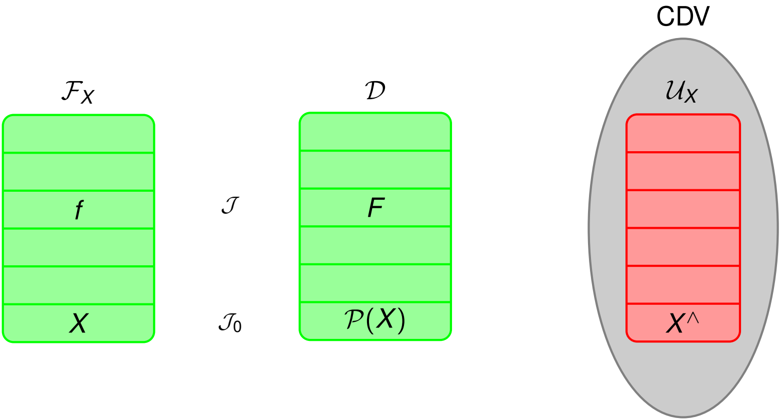

Logical Relations

- $\mathcal{J}_o = \{(f, F) \mid f \in F\}$, auxiliary relation $\mathcal{I}_o = \{(f, \{f\}) \mid f \in X\}$
- $\mathcal{J}$ = logical relation induced by $\mathcal{J}_o$ and $\mathcal{I}$ = logical relation induced by $\mathcal{I}_o$

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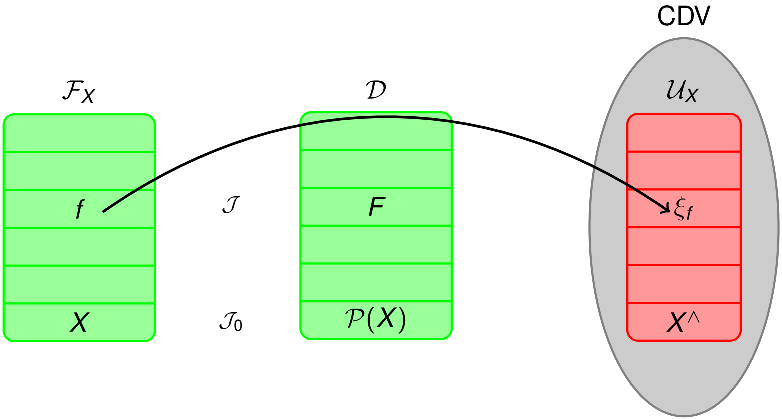

Every $f \in \mathcal{F}_X(A)$ represents a $\xi_f \in \mathcal{U}_X(A)$

- $A = 0$, then $\xi_f = f$,
- $A = B \to C$, then $\xi_f = \bigwedge_{g \in \mathcal{F}_X(B)} \xi_g \to \xi_{fg}$.

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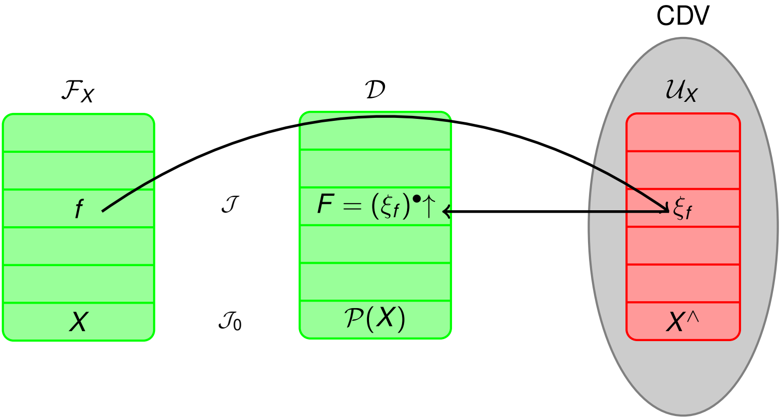
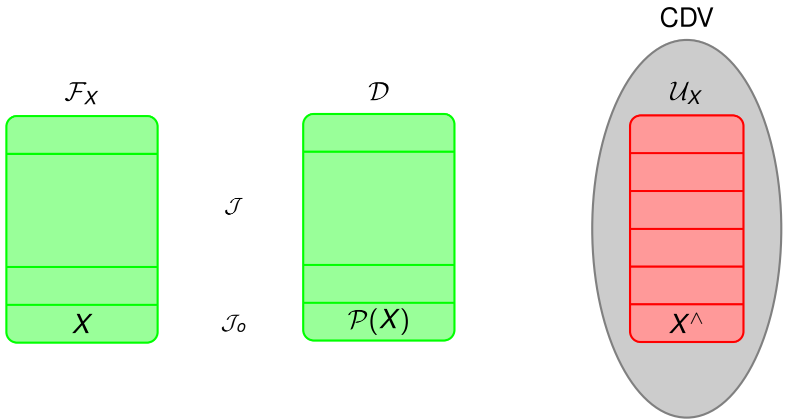


Idea: the construction "factorize"!

Ready to go: $[\![?]\!] = f \in \mathcal{F}_X$



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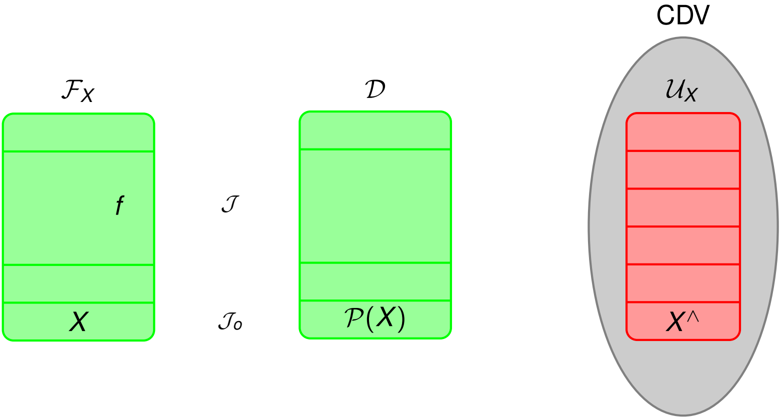

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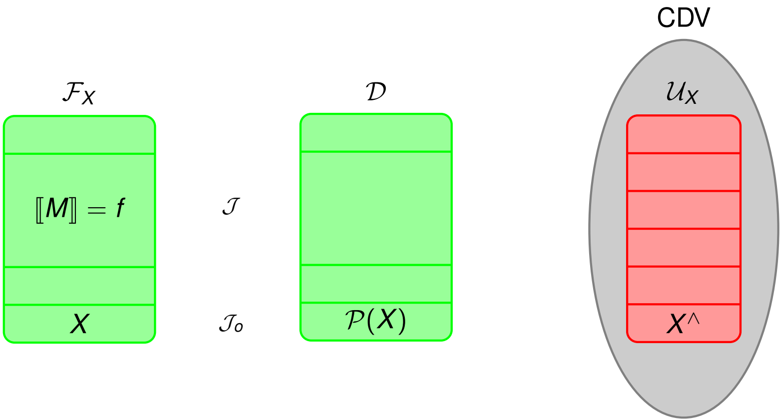

$[\![M]\!]^{\mathcal{F}} = f$

Focus on M simply typable and normal.

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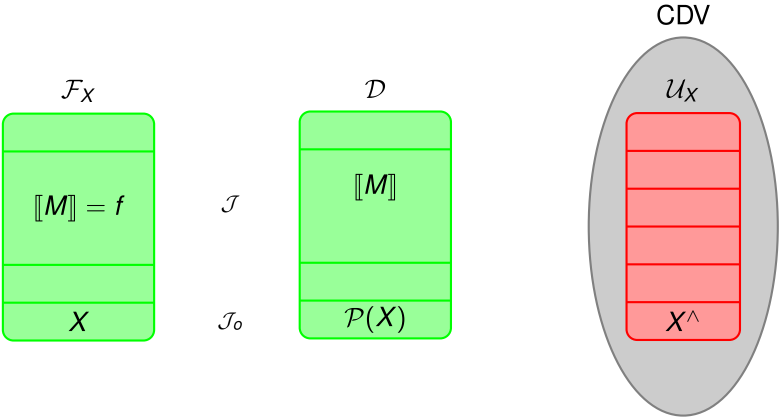

$$[\![M]\!]^{\mathcal{F}} = f \quad \iff \quad f \; \mathcal{J} \; [\![M]\!]^{\mathcal{D}}$$

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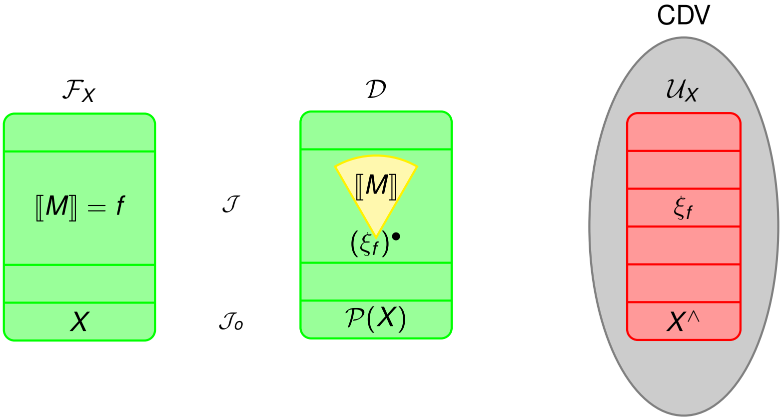
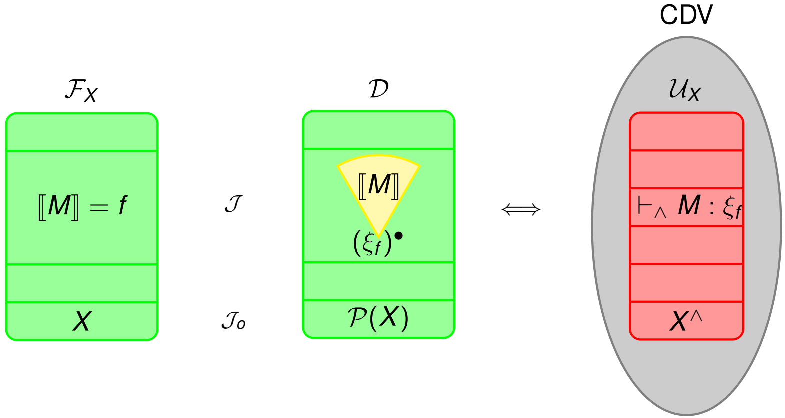


$$[\![M]\!]^{\mathcal{F}} = f \iff f \,\mathcal{J}\, [\![M]\!]^{\mathcal{D}} \iff [\![M]\!]^{\mathcal{D}} \in (\xi_f)^{\bullet}\!\uparrow$$

Focus on M simply typable and normal.

Ready to go: $[\![{\color{red}?}]\!] = f \in \mathcal{F}_X$



$$[\![M]\!]^{\mathcal{F}} = f \quad \iff \quad f\ \mathcal{J}\ [\![M]\!]^{\mathcal{D}} \quad \iff \quad [\![M]\!]^{\mathcal{D}} \in (\xi_f)^\bullet\!\uparrow \quad \iff \quad \vdash_\wedge M : \xi_f$$

If IHP_n for (Uniform) Intersection Types is decidable, then λ-definability in $\mathcal{F}_n$ is decidable $\lightning$ (for $n > 1$ by Joly)

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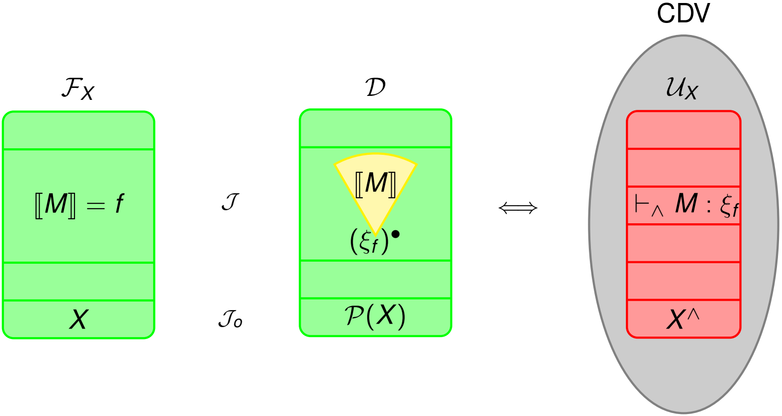

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$\mathrm{DP}_n \leq_T \mathrm{IHP}_n$

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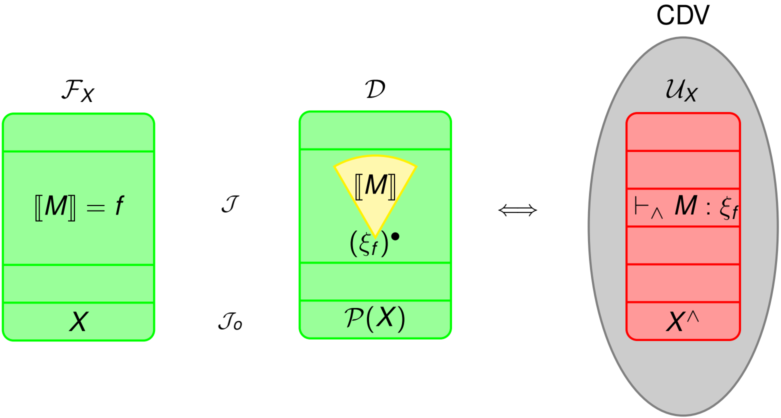

$$[\![M]\!]^{\mathcal{F}} = f \quad \Longleftrightarrow \quad f \; \mathcal{J} \; [\![M]\!]^{\mathcal{D}} \quad \Longleftrightarrow \quad [\![M]\!]^{\mathcal{D}} \in (\xi_f)^\bullet\!\uparrow \quad \Longleftrightarrow \quad \vdash_\wedge M : \xi_f$$

Definability Problem $\leq_T$ Inhabitation Problem for CDV

Concluding Remarks

Refinement of Urzyczyin's Result

IHP_n is undecidable for $n > 1$.

Degrees of Reduction

For Uniform Types:

- Inhabitation Problem $\leq_T$ Definability Problem (proper Turing-reduction)
- Definability Problem $\leq_T$ Inhabitation Problem (many-one reduction)

Logically simpler!

There exists a total computable function ϕ such that $IHP = \phi^{-1}(DP)$.

Are DP and IHP many-one equivalent?

What about non-uniform types?

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Thanks for your attention!