AN INTERACTIVE SEMANTICS FOR CLASSICAL PROOFS

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September 26, 2013

Introduction

In this talk ... (I)

- I will introduce a language consisting of **infinitary propositional formulas** and a **sequent calculus** to derive *sequences* of formulas $\Theta, \Gamma, \Delta \ldots$.
- I will show a **soundness**–and–**completeness** theorem for **derivations** = formal proofs in the sequent calculus:

In this talk ... (I)

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- I will show a **soundness**–and–**completeness** theorem for **derivations** = formal proofs in the sequent calculus:

Soundness :	$\mathfrak{T} \in \mathbf{DER}(\mathbf{\Theta})$	$\Longrightarrow$	$\mathfrak{T} \in \mathbf{INT}(\mathbf{\Theta})$
Completeness :	$\mathfrak{T} \in \mathbf{INT}(\mathbf{\Theta})$	$\Longrightarrow$	$\mathfrak{T} \in \mathbf{DER}(\mathbf{\Theta})$

where, roughly speaking:

- $\mathfrak{T} \in \mathbf{DER}(\mathbf{\Theta})$ = "$\mathfrak{T}$ is the **skeleton** of a *derivation* of $\mathbf{\Theta}$"
- $\mathfrak{T} \in \mathbf{INT}(\mathbf{\Theta})$ = "$\mathfrak{T}$ is a member of the *interactive interpretation* of $\mathbf{\Theta}$"

Skeleton: example

Untyped λ–terms: $t, u \ldots ::= x, y, z, \ldots \mid tu \mid \lambda x.t$
Formulas (types): $F, G, \ldots ::= a, c, d, \ldots \mid F \to G$

Curry–style type assignment for untyped λ–terms:

$$\Gamma, x : F \vdash x : F \qquad \frac{\Gamma, x : F \vdash t : G}{\Gamma \vdash \lambda x.t : F \to G}$$

$$\frac{\Gamma \vdash t : F \to G \qquad \Gamma \vdash u : F}{\Gamma \vdash tu : G}$$

Skeleton: example

Untyped λ–terms: $t, u \ldots ::= x, y, z, \ldots \mid tu \mid \lambda x.t$
Formulas (types): $F, G, \ldots ::= a, c, d, \ldots \mid F \to G$

Derivations in minimal logic (natural deduction):

$$\Gamma, F \vdash F \qquad \frac{\Gamma, F \vdash G}{\Gamma \vdash F \to G}$$

$$\frac{\Gamma \vdash F \to G \qquad \Gamma \vdash F}{\Gamma \vdash G}$$

Untyped λ–terms $\approx$ skeletons of derivations in *natural deduction* of formulas of *minimal logic*.

- But this is just an intuition. In our case, skeletons are **infinitary** (not even well–founded) objects.

In this talk ... (II)

- The *interactive interpretation* of a sequent Θ is defined ... **interactively** = using a procedure of **cut–elimination**

- Related works: **Girard's ludics** and (in part) **Krivine's classical realizability**.

J.–Y. Girard
Locus solum: From the rules of logic to the logic of rules
Math. Struct. in Comp. Sci. 11(3) 301–506, 2001.

J.–L. Krivine
Realizability in classical logic
Panoramas et Synthèses 27 197–229, 2009.

This work is self–contained = **no** familiarity is required.

Analogies and differences

	Ludics	Realizability	This work
Logic	$\mathbf{MALL2}_{foc}$	Analysis	$\mathcal{T}$
Object = untyped proof	design	λ_c–term	test
Counter–object	design	stack	environment
Polarization	YES	NO	NO
Interaction (W)	cut–net	process	configuration
Interaction (H)	cut–elimination	**KAM**	**closed** cut–elimination
Orthogonality	convergence	several options	convergence
Formula	behaviour	truth value	interactive interpretation
Aim	soundness correctness completeness	adequacy extraction	soundness completeness

- $\mathbf{MALL2}_{foc}$ = second–order polarized focalized **MALL**, sequent calculus **with cut**
- Analysis = second–order classical arithmetic, natural deduction
- $\mathcal{T}$ = $\mathcal{T}$ait calculus (only *normal* rules) **cut–free** sequent calculus
- **KAM** = **K**rivine **A**btract **M**achine (head β–reduction)
- Behaviour = a set of designs **A** such that $\mathbf{A} = \mathbf{A}^{\perp\perp}$
- Truth value = a set of closed λ_c–terms **A** such that $\mathbf{A} = \mathbf{X}^{\perp}$, for some set of stacks **X**
- Interactive interpretation = a set of tests **A** such that $\mathbf{A} = \{\mathbf{E}\}^{\perp}$, for some environment **E**
- Correctness = the interpretation is invariant under cut–elimination

The **interactive interpretation** of formulas and sequents is defined through the concept of ... **interaction**.

We now informally describe what interaction is.

Sequent calculus

- **Formulas**: $\mathbf{F}, \mathbf{G}, \mathbf{H}, \ldots$ generated in the usual way, using (possibly infinitary) connectives $\vee, \wedge, ^{\perp}$.
- **Sequents** : $\mathbf{\Theta}, \mathbf{\Gamma}, \ldots =$ finite, non–empty sequences of formulas $\vdash \mathbf{F}_0, \ldots, \mathbf{F}_{n-1}$.
- **Rules** for deriving sequents.

$$\frac{\{\mathbf{\Theta}_a\}_{a \in S}}{\mathbf{\Theta}}\ (\mathrm{r})$$

- **Derivations** = well–founded trees labeled by sequents (which are “locally correct”).

$$\text{System } \mathcal{T} \overset{\text{DEF}}{=} (\mathbf{F}, \mathbf{S}, \mathbf{R}, \mathbf{D})$$

Closed cuts

$$\dfrac{\begin{array}{c}\vdots\,\pi\\ \vdash \mathbf{F}_0,\ldots,\mathbf{F}_{n-1}\end{array}\qquad \mathbf{G}_0,\ldots,\mathbf{G}_{n-1}}{}\ \textbf{cut}$$

where:

- "Closed" means that every formula is a cut–formula.
- π is a **derivation** of $\vdash \mathbf{F}_0,\ldots,\mathbf{F}_{n-1}$ in $\mathcal{T}$,
- $\mathbf{G}_0,\ldots,\mathbf{G}_{n-1}$ is a finite, non–empty sequence of formulas of $\mathcal{T}$ that we call **environment**.
- We simultaneously cut $\mathbf{F}_i$ with $\mathbf{G}_i$, for each $i < n$.

In this talk, we adopt the more suggestive notation:

$$\dfrac{\begin{array}{c}\vdots\,\pi\\ \vdash \mathbf{F}_0,\ldots,\mathbf{F}_{n-1}\end{array}\qquad \begin{array}{c}\vdots\\ \vdash_* \mathbf{G}_0\end{array}\ \ \ldots\ \ \begin{array}{c}\vdots\\ \vdash_* \mathbf{G}_{n-1}\end{array}}{}\ \textbf{cut}$$

We also denote environments by $\vdash_* \mathbf{G}_0 \ \ \ldots\ \ \vdash_* \mathbf{G}_{n-1}$.

Interaction (I)

So ... you cut a **derivation** AND a **sequence of formulas** ???

Interaction (I)

So ... you cut a **derivation** AND a **sequence of formulas** ???

YES, because we identify a formula **F** occurring in $\mathfrak{E}$ with the **derivation** of its **subformula tree**:

$$\dfrac{\begin{matrix}\vdots \\ \vdash_* \mathbf{F}\end{matrix} \qquad \begin{matrix}\vdots \\ \vdash_* \mathbf{G}\end{matrix} \qquad \begin{matrix}\vdots \\ \vdash_* \mathbf{H}\end{matrix}}{\vdash_* \mathbf{F} \wedge \mathbf{G} \wedge \mathbf{H}} \qquad \dfrac{\begin{matrix}\vdots \\ \vdash_* \mathbf{F}\end{matrix} \qquad \begin{matrix}\vdots \\ \vdash_* \mathbf{G}\end{matrix} \qquad \begin{matrix}\vdots \\ \vdash_* \mathbf{H}\end{matrix}}{\vdash_* \mathbf{F} \vee \mathbf{G} \vee \mathbf{H}}$$

This **special kind of derivation** is not in general a **derivation** in $\mathcal{T}$. Since every formula has a **unique subformula tree**, every formula **has a unique special derivation** of this kind. Moreover, it makes sense to cut these thing together ...

Interaction (II)

…and define a **procedure** of cut–elimination:

$$
\dfrac{
  \dfrac{\dfrac{\vdots\,\pi}{\vdash\ \mathbf{F} \vee \mathbf{G}, \mathbf{F}}}{\vdash\ \mathbf{F} \vee \mathbf{G}}
  \qquad
  \dfrac{\dfrac{\vdots}{\vdash_{*}\ \mathbf{F}^{\perp}} \qquad \dfrac{\vdots}{\vdash_{*}\ \mathbf{G}^{\perp}}}{\vdash_{*}\ \mathbf{F}^{\perp} \wedge \mathbf{G}^{\perp}}
}{}\ \mathbf{cut}
$$

reduces to

$$
\dfrac{
  \dfrac{\vdots\,\pi}{\vdash\ \mathbf{F} \vee \mathbf{G}, \mathbf{F}}
  \qquad
  \dfrac{\dfrac{\vdots}{\vdash_{*}\ \mathbf{F}^{\perp}} \qquad \dfrac{\vdots}{\vdash_{*}\ \mathbf{G}^{\perp}}}{\vdash_{*}\ \mathbf{F}^{\perp} \wedge \mathbf{G}^{\perp}}
  \qquad
  \dfrac{\vdots}{\vdash_{*}\ \mathbf{F}^{\perp}}
}{}\ \mathbf{cut}
$$

Interaction (II)

. . . and define a **procedure** of cut–elimination:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi\\ \vdash\ \mathbf{F}\vee\mathbf{G},\mathbf{F}\end{array}}{\vdash\ \mathbf{F}\vee\mathbf{G}} \qquad \dfrac{\begin{array}{c}\vdots\\ \vdash_* \mathbf{F}^\perp\end{array} \qquad \begin{array}{c}\vdots\\ \vdash_* \mathbf{G}^\perp\end{array}}{\vdash_* \mathbf{F}^\perp\wedge\mathbf{G}^\perp}}{}\ \mathbf{cut}$$

reduces to

$$\dfrac{\begin{array}{c}\vdots\ \pi\\ \vdash \mathbf{F}\vee\mathbf{G},\mathbf{F}\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ \vdash_* \mathbf{F}^\perp\end{array} \qquad \begin{array}{c}\vdots\\ \vdash_* \mathbf{G}^\perp\end{array}}{\vdash_* \mathbf{F}^\perp\wedge\mathbf{G}^\perp} \qquad \begin{array}{c}\vdots\\ \vdash_* \mathbf{F}^\perp\end{array}}{}\ \mathbf{cut}$$

This form of cut–elimination does not produce anything.
However, we can study some **properties of this procedure**.

Interaction (III)

In general, we have to consider closed cuts like

$$\dfrac{\dfrac{\vdots\,\pi}{\vdash \mathbf{F}_0, \ldots, \mathbf{F}_{n-1}} \qquad \dfrac{\vdots}{\vdash_* \mathbf{G}_0} \quad \ldots \quad \dfrac{\vdots}{\vdash_* \mathbf{G}_{n-1}}}{} \,\textbf{cut}$$

There are new situations to consider:

- **Error**:

$$\dfrac{\dfrac{\dfrac{\vdots\,\pi}{\vdash\ \mathbf{F}_1 \vee \mathbf{F}_2, \mathbf{F}_1}}{\vdash\ \mathbf{F}_1 \vee \mathbf{F}_2} \qquad \dfrac{\vdots\,\pi}{\vdash_* \mathbf{G}_1 \vee \mathbf{G}_2}}{} \,\textbf{cut}$$

  reduces to "**error**."

Interaction (IV)

- **Reduction**:

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi \\ \vdash\ \mathbf{F}_1 \lor \mathbf{F}_2, \mathbf{F}_1\end{array}}{\vdash\ \mathbf{F}_1 \lor \mathbf{F}_2} \qquad \dfrac{\begin{array}{c}\vdots \\ \vdash_* \mathbf{G}_1\end{array} \quad \begin{array}{c}\vdots \\ \vdash_* \mathbf{G}_2\end{array} \quad \begin{array}{c}\vdots \\ \vdash_* \mathbf{G}_3\end{array}}{\vdash_* \mathbf{G}_1 \land \mathbf{G}_2 \land \mathbf{G}_3}}{} \ \mathbf{cut}
$$

reduces to

$$
\dfrac{\begin{array}{c}\vdots\,\pi \\ \vdash \mathbf{F}_1 \lor \mathbf{F}_2, \mathbf{F}_1\end{array} \qquad \dfrac{\begin{array}{c}\vdots \\ \vdash_* \mathbf{G}_1\end{array} \quad \begin{array}{c}\vdots \\ \vdash_* \mathbf{G}_2\end{array} \quad \begin{array}{c}\vdots \\ \vdash_* \mathbf{G}_3\end{array}}{\vdash_* \mathbf{G}_1 \land \mathbf{G}_2 \land \mathbf{G}_3} \qquad \begin{array}{c}\vdots \\ \vdash_* \mathbf{G}_1\end{array}}{} \ \mathbf{cut}
$$

Generalization

- Instead of considering **derivations** in $\mathcal{T}$, we will consider **skeletons of derivations**, that we call **tests** $\mathfrak{T}, \mathfrak{U}, \mathfrak{V}, \ldots$
- A test does **not contain all** the information of a derivation. But we have enough information to consider closed cuts of the form

$$\dfrac{\mathfrak{T} \qquad \begin{array}{c}\vdots \\ \vdash_* \mathbf{G}_0\end{array} \quad \ldots \quad \begin{array}{c}\vdots \\ \vdash_* \mathbf{G}_{n-1}\end{array}}{} \ \mathbf{cut}$$

where we cut:
  - the **test** $\mathfrak{T}$,
  - the **environment** $\vdash_* \mathbf{G}_0 \quad \ldots \quad \vdash_* \mathbf{G}_{n-1}$.

Later on, we will define a suitable procedure of reduction (**cut–elimination**) that we call **interaction**.

Trees

Notation

- $\mathbb{N}^* = \{s, t, u, \ldots\}$ = the set of **finite sequences of natural numbers.**
- Some sequences:

$$\begin{array}{rcl} (\,) & = & \text{the } \textbf{empty sequence}; \\ a & = & \text{unary sequence}; \\ a_0 a_1 & = & \text{binary sequence}; \\ a_0 a_1 \cdots a_{k-1} & = & k\text{–ary sequence}. \end{array}$$

- st = the **concatenation** of s and t.
- In particular, if s is a k–ary sequence and $a \in \mathbb{N}$, then sa and sa are $(k+1)$–ary sequences.
- **Prefix order**: $s \sqsubseteq t \overset{\text{DEF}}{\Longleftrightarrow}$ there is $u \in \mathbb{N}^*$ such that $t = su$.

Trees

- A **tree** T is a non–empty subset of $\mathbb{N}^*$ such that
$$\text{if } t \in T \text{ and } s \sqsubseteq t, \text{ then } s \in T.$$
- Since T is non–empty, $(\,) \in T$. $(\,)$ is called the **root** of T.
- An **infinite branch** in T is a infinite subset $S \subseteq T$ of the form $S = \{(\,)\ ,\ a_0\ ,\ a_0 a_1\ ,\ \dots\ ,\ a_0 a_1 \cdots a_{n-1}\ ,\ \dots\}$.
- A tree is said to be **well–founded** if it does not contain an infinite branch.
- Let A be a non–empty set. A **tree labeled by A** is a pair $L = (T, \varphi)$ consisting of a tree T and a function $\varphi : T \longrightarrow A$. φ is called the **labeling function** of L. A is called the set of **labels**.
- We write $\mathrm{TREE}(L)$ and $\mathrm{LAB}(L)$ for the underlying tree of L and its labeling function respectively, i.e., if $L = (T, \varphi)$, then $\mathrm{TREE}(L) = T$ and $\mathrm{LAB}(L) = \varphi$.
- Two labeled trees L and M (labeled by the *same* set of labels) are **equal** if $\mathrm{TREE}(L) = \mathrm{TREE}(M)$ and $\mathrm{LAB}(L)(s) = \mathrm{LAB}(M)(s)$, for all $s \in \mathrm{TREE}(L)$.

Tait calculus $\mathcal{T}$

W.W. Tait
Normal derivability in classical logic
In: The syntax and semantics of infinitary languages (Jon Barwise editor), LNM **72** Springer–Verlag 204–236, 1968.

H. Schwichtenberg
Proof theory: some applications of cut-elimination
In: Handbook of Mathematical Logic (Jon Barwise editor) 867–895, 1977.

W.W. Tait
Gödel's reformulation of Gentzen's first consistency proof for arithmetic: the no-counterexample interpretation
The Bulletin of Symbolic Logic **11**(2) 225-238, 2005.

W. Pohlers
Proof theory: an introduction
Spinger–Verlag 1989.

Tait calculus is an **infinitary classical** propositional logic.

- A *purely logical* and propositional approach to (first order, classical) arithmetic.

In this work:

- Sequents are **finite sequences of formulas** rather than **finite sets of formulas**,
- We only consider **sets of natural numbers** as index sets.
- We do not consider propositional atoms: the prime (i.e., undecomposable) formulas are **0** (false) and **1** (true).
- We only consider **normal rules** (i.e., no cut–rule).

Formulas

- The **formulas** of our language are inductively defined as follows:

if for some $S \subseteq \mathbb{N}$, $\{\mathbf{G}_a\}_{a \in S}$ is a family of formulas, then $\bigvee_S \mathbf{G}_a$ and $\bigwedge_S \mathbf{G}_a$ are formulas.

Some terminology and notation:

- $\bigvee_S \mathbf{G}_a$ = **disjunction**;
- $\bigwedge_S \mathbf{G}_a$ = **conjunction**;
- $\mathbf{0} \stackrel{\text{DEF}}{=} \bigvee_\emptyset \mathbf{G}_a$;
- $\mathbf{1} \stackrel{\text{DEF}}{=} \bigwedge_\emptyset \mathbf{G}_a$.

Equilvalently, a **formula** is a well–founded tree labeled by $\{\vee, \wedge\}$.

Negation and sequents

The **negation** of a formula $\mathbf{F}$, noted by $\mathbf{F}^{\perp}$, is the formula recursively defined as follows:

$$(\bigvee_S \mathbf{G}_a)^{\perp} \stackrel{\text{DEF}}{=} \bigwedge_S (\mathbf{G}_a^{\perp}); \qquad (\bigwedge_S \mathbf{G}_a)^{\perp} \stackrel{\text{DEF}}{=} \bigvee_S (\mathbf{G}_a^{\perp}).$$

In particular, $\mathbf{0}^{\perp} = \mathbf{1}$, and $\mathbf{1}^{\perp} = \mathbf{0}$.

The negation is **involutive**:

$$\mathbf{F}^{\perp\perp} = \mathbf{F}.$$

A **sequent** $\Theta, \Gamma, \ldots$ of $\mathcal{T}$ is a non–empty finite sequence $\vdash \mathbf{F}_0, \ldots, \mathbf{F}_{n-1}$ of formulas ($n > 0$).

Rules

The following **rules** derive *sequents*. They have to be read bottom–up, in the sense of *proof–search*.

Disjunctive rule :

- $i < n$ and $a_0 \in S$:

$$\frac{\vdash \ \mathbf{F}_0, \ldots, \mathbf{F}_{i-1} \ , \ \bigvee_S \mathbf{G}_a \ , \ \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{n-1} \ , \ \mathbf{G}_{a_0}}{\vdash \ \mathbf{F}_0, \ldots, \mathbf{F}_{i-1} \ , \ \bigvee_S \mathbf{G}_a \ , \ \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{n-1}} \ (\vee)$$

Conjunctive rule :

- $i < n$, one premise for each member of S:

$$\frac{\vdash \ \mathbf{F}_0, \ldots, \mathbf{F}_{i-1} \ , \ \bigwedge_S \mathbf{G}_a \ , \ \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{n-1} \ , \ \mathbf{G}_a \qquad \ldots \text{all } a \in S}{\vdash \ \mathbf{F}_0, \ldots, \mathbf{F}_{i-1} \ , \ \bigwedge_S \mathbf{G}_a \ , \ \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{n-1}} \ (\wedge)$$

Derivations

- A **derivation** is a **well–founded tree labeled by sequents** which is "locally correct." Formally,

A **derivation** is a **well–founded tree** π **labeled by sequents** such that for all $s \in \mathtt{TREE}(\pi)$ one of the following two conditions holds:

$$(\mathbf{D}_1):\begin{cases}\text{(i)} & \mathtt{LAB}(\pi)(s) = \vdash \mathbf{F}_0,\ldots,\mathbf{F}_{n-1} \text{ and there are } i<n \\ & \text{and } a_0\in\mathbb{N} \text{ such that } \mathbf{F}_i = \bigvee_S \mathbf{G}_a \text{ and } a_0\in S, \\ \text{(ii)} & sa\in\mathtt{TREE}(\pi) \text{ if and only if } a=0, \text{ and} \\ & \mathtt{LAB}(\pi)(s0) = \vdash \mathbf{F}_0,\ldots,\mathbf{F}_{n-1},\mathbf{G}_{a_0}.\end{cases}$$

$$(\mathbf{D}_2):\begin{cases}\text{(i)} & \mathtt{LAB}(\pi)(s) = \vdash \mathbf{F}_0,\ldots,\mathbf{F}_{n-1} \text{ and there is } i<n \\ & \text{such that } \mathbf{F}_i = \bigwedge_S \mathbf{G}_a, \\ \text{(ii)} & sa\in\mathtt{TREE}(\pi) \text{ if and only if } a\in S, \text{ and} \\ & \mathtt{LAB}(\pi)(sa) = \vdash \mathbf{F}_0,\ldots,\mathbf{F}_{n-1},\mathbf{G}_a, \text{ for all } a\in S.\end{cases}$$

This completes the definition of **Tait calculus** $\mathcal{T}$.

Some derivable sequents

- **Initial sequents** : A derivation with no premises is

$$\frac{}{\vdash\ \mathbf{F}_0, \ldots, \mathbf{F}_{i-1}\ ,\ \mathbf{1}\ ,\ \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{n-1}}\ (\wedge)$$

  - Every leaf of a derivation is labeled by a sequent of this form.

- **Generalized identities** : Sequents of this form are derivable:

$$\vdash\ \mathbf{F}_0, \ldots, \mathbf{F}_{i-1}\ ,\ \mathbf{G}\ ,\ \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{j-1}\ ,\ \mathbf{G}^{\perp}\ ,\ \mathbf{F}_{j+1}, \ldots, \mathbf{F}_{n-1}$$

- **Novikoff's law of complete induction** is the formula

$$\big(F_1 \wedge (F_1 \to F_2) \wedge (F_2 \to F_3) \wedge \cdots\big) \to F_1 \wedge F_2 \wedge F_3 \wedge \cdots .$$

  In our system, we can consider the sequent

$$\vdash\ \big(\mathbf{F}_1^{\perp} \vee (\mathbf{F}_1 \wedge \mathbf{F}_2^{\perp}) \vee (\mathbf{F}_2 \wedge \mathbf{F}_3^{\perp}) \vee \cdots\big)\ ,\ \mathbf{F}_1 \wedge \mathbf{F}_2 \wedge \mathbf{F}_3 \wedge \cdots .$$

  and show that it is derivable.

Game interpretation (I)

We can give a game–theoretic interpretation of our sequent calculus derivations (Tait (2005)). The game is played by two participants: **SHE** and **HE**. They argue about some sequent $\boldsymbol{\Theta}$. **SHE** tries to prove it, whereas **HE** tries to refute it.
A **play for** $\boldsymbol{\Theta}$ proceeds as follows.

- The play starts with $\boldsymbol{\Theta}_0 \stackrel{\text{DEF}}{=} \boldsymbol{\Theta}$.

Let $\boldsymbol{\Theta}_k \ = \vdash \mathbf{F}_0, \ldots, \mathbf{F}_{n-1}$.

- If $\boldsymbol{\Theta}_k$ only contains occurrences of prime formulas **0** and **1**, then $\boldsymbol{\Theta}_{k+1} \stackrel{\text{DEF}}{=} \boldsymbol{\Theta}_k$.
- Otherwise, **SHE** selects an occurrence of non–prime formula, say $\mathbf{F}_i$.
  If $\mathbf{F}_i$ is a **disjunctive** formula $\bigvee_S \mathbf{G}_a$, then **SHE** chooses $a_0 \in S$ and $\boldsymbol{\Theta}_{k+1} \stackrel{\text{DEF}}{=} \vdash \mathbf{F}_0, \ldots, \mathbf{F}_{n-1}, \mathbf{G}_{a_0}$.
  If $\mathbf{F}_i$ is a **conjunctive** formula $\bigwedge_S \mathbf{G}_a$, then **HE** chooses $a_0 \in S$ and $\boldsymbol{\Theta}_{k+1} \stackrel{\text{DEF}}{=} \vdash \mathbf{F}_0, \ldots, \mathbf{F}_{n-1}, \mathbf{G}_{a_0}$.

Game interpretation (II)

SHE wins the play if for some n the sequent Θ_n contains some occurrences of $\mathbf{1}$. Otherwise, **HE** wins.

This game is clearly unfair to . . . **HIM**:

- **HE** can only choose an immediate subformula of a **conjunctive** formula selected by **HER**.
- **SHE** can choose any occurence of non–prime formula in a sequent, and in case it is **disjunctive**, any immediate subformula of it. In particular, **if SHE realizes that a previous choice was wrong, then SHE can remedy later on, making a different choice.**

SHE has a **strategy** to win all the possible plays for Θ if and only if Θ is derivable in $\mathcal{T}$.

Tests

Actions

Tests $\approx$ skeletons of derivations in $\mathcal{T}$.

Formally, tests are **infinitary trees** labeled by **actions**.

- A **disjunctive action** is a triple $\langle n, i, a \rangle$ where n, i, a are natural numbers and $i < n$.
- A **conjunctive action** is a pair $[n, i]$ where n, i are natural numbers and $i < n$.

Some terminology:

- $\langle n, i, a \rangle = \langle$ **base** , **address** , **name** $\rangle$
- $[n, i] \quad = [$ **base** , **address** $]$

Tests

A **test** is a **tree labeled by actions** $\mathfrak{T}$ such that for all $s \in \texttt{TREE}(\mathfrak{T})$ one of the following two conditions holds:

$$\textbf{(T}_1\textbf{)} : \begin{cases} \text{(i)} & \texttt{LAB}(\mathfrak{T})(s) = \langle n, i, a_0 \rangle, \\ \text{(ii)} & sa \in \texttt{TREE}(\mathfrak{T}) \text{ if and only if } a = 0, \text{ and} \\ & \text{the base of } \texttt{LAB}(\mathfrak{T})(s0) \text{ is } n+1. \end{cases}$$

$$\textbf{(T}_2\textbf{)} : \begin{cases} \text{(i)} & \texttt{LAB}(\mathfrak{T})(s) = [n, i], \\ \text{(ii)} & \text{for all } a \in \mathbb{N},\ sa \in \texttt{TREE}(\mathfrak{T}) \text{ and} \\ & \text{the base of } \texttt{LAB}(\mathfrak{T})(sa) \text{ is } n+1. \end{cases}$$

We use letters $\mathfrak{T}, \mathfrak{U}, \mathfrak{V}, \ldots$ to range over tests.

- Tests are **not** well–founded trees.

Terminology and notation

Let $\mathfrak{T}$ be a test.

- If the base of the action $\texttt{LAB}(\mathfrak{T})(())$ is n, we say that $\mathfrak{T}$ **is on base** n.
- If $\texttt{LAB}(\mathfrak{T})(()) = \langle n, i, a_0 \rangle$, then we say that $\mathfrak{T}$ is a **disjunctive test**. By definition, $\mathfrak{T}$ has a unique immediate subtree $\mathfrak{U}$. We denote $\mathfrak{T}$ by

$$\langle n, i, a_0 \rangle . \mathfrak{U}$$

- If $\texttt{LAB}(\mathfrak{T})(()) = [n, i]$, then we say that $\mathfrak{T}$ is a **conjunctive test**. By definition, for each $a \in \mathbb{N}$ there is an immediate subtree $\mathfrak{U}_a$ of $\mathfrak{T}$. We denote $\mathfrak{T}$ by

$$[n, i] . \mathfrak{U}_a$$

Example

$$\mathfrak{T} \stackrel{\text{DEF}}{=} \langle 1, 0, a_0 \rangle . \langle 2, 0, a_0 \rangle \ldots \langle n, 0, a_0 \rangle . \langle n+1, 0, a_0 \rangle \ldots$$

is a disjunctive test on base 1. Here:

- $\mathtt{TREE}(\mathfrak{T}) = \{(), 0, 00, 000, \ldots\} = \{0^n \mid n \in \mathbb{N}\}$,
- $\mathtt{LAB}(\mathfrak{T})(0^n) = \langle n+1, 0, a_0 \rangle$, for each $n \in \mathbb{N}$.

DER(Θ)

Inductive definition of **DER**(**Θ**):

$$\langle n, i, a_0\rangle.\mathfrak{U} \in \mathbf{DER}(\vdash \mathbf{F}_0,\ldots,\mathbf{F}_{i-1}, \bigvee_S \mathbf{G}_a, \mathbf{F}_{i+1},\ldots,\mathbf{F}_{n-1}) \dfrac{\mathfrak{U} \in \mathbf{DER}(\vdash \mathbf{F}_0,\ldots,\mathbf{F}_{i-1}, \bigvee_S \mathbf{G}_a, \mathbf{F}_{i+1},\ldots,\mathbf{F}_{n-1}, \mathbf{G}_{a_0})}{} \; (\vee)$$

$$\dfrac{\mathfrak{U} \in \mathbf{DER}(\vdash \mathbf{F}_0,\ldots,\mathbf{F}_{i-1}, \bigvee_S \mathbf{G}_a, \mathbf{F}_{i+1},\ldots,\mathbf{F}_{n-1}, \mathbf{G}_{a_0})}{\langle n, i, a_0\rangle.\mathfrak{U} \in \mathbf{DER}(\vdash \mathbf{F}_0,\ldots,\mathbf{F}_{i-1}, \bigvee_S \mathbf{G}_a, \mathbf{F}_{i+1},\ldots,\mathbf{F}_{n-1})} \; (\vee)$$

$$\dfrac{\mathfrak{U}_a \in \mathbf{DER}(\vdash \mathbf{F}_0,\ldots,\mathbf{F}_{i-1}, \bigwedge_S \mathbf{G}_a, \mathbf{F}_{i+1},\ldots,\mathbf{F}_{n-1}, \mathbf{G}_a) \;\ldots \text{all } a \in S}{[n, i].\mathfrak{U}_a \in \mathbf{DER}(\vdash \mathbf{F}_0,\ldots,\mathbf{F}_{i-1}, \bigwedge_S \mathbf{G}_a, \mathbf{F}_{i+1},\ldots,\mathbf{F}_{n-1})} \; (\wedge)$$

In the conjunctive rule the subtests $\{\mathfrak{U}_b\}_{b \in \mathbb{N} \setminus S}$ are arbitrary. For instance, we have **DER**($\vdash$ **1**) = conjunctive tests on base 1.

Remarks

- There is **no bijective correspondence** between

  $$\mathbf{DER}(\boldsymbol{\Theta}) \quad \text{and} \quad \{\pi : \pi \text{ is a derivation of } \boldsymbol{\Theta} \text{ in } \mathcal{T}\}.$$

  For instance, the sequent $\vdash \mathbf{1}$ has **exactly one** derivation in $\mathcal{T}$, but $\mathbf{DER}(\vdash \mathbf{1}) =$ conjunctive tests on base 1.
- The set $\mathbf{DER}(\boldsymbol{\Theta})$ is defined **syntactically**, i.e., by using the rules of the sequent calculus.
- Our aim now is to define the set $\mathbf{INT}(\boldsymbol{\Theta})$ **interactively**, i.e., by using a kind of cut–elimination procedure.

Interaction

Recall that we want to consider **closed cuts** of the form

$$\frac{\mathfrak{T} \qquad \vdash_* \mathbf{G}_0 \quad \ldots \quad \vdash_* \mathbf{G}_{n-1}}{} \ \textbf{cut}$$

and define a suitable procedure of reduction (**cut–elimination**) that we call **interaction**. Here:

- $\mathfrak{T}$ is a **test**,
- $\vdash_* \mathbf{G}_0 \quad \ldots \quad \vdash_* \mathbf{G}_{n-1}$ is an **environment**, that is a **sequence of formulas** (recall that we identify an occurrence of formula in $\mathfrak{E}$ with the derivation of its subformula tree)

$$\begin{array}{ccc} \vdots & & \vdots \\ \vdash_* \mathbf{G}_0 & \ldots & \vdash_* \mathbf{G}_{n-1} \end{array}$$

Configurations

An **environment on base** n ($n > 0$) is a sequence of formulas $\mathbf{G}_0, \ldots, \mathbf{G}_{n-1}$ that we denote by $\vdash_* \mathbf{G}_0 \quad \ldots \quad \vdash_* \mathbf{G}_{n-1}$.

A **configuration** is either

- a pair $\left(\mathcal{T}, \vdash_* \mathbf{G}_0, \ldots, \vdash_* \mathbf{G}_{n-1}\right)$ where:
  - $\mathcal{T}$ is a **test** of base n,
  - $\vdash_* \mathbf{G}_0, \ldots, \vdash_* \mathbf{G}_{n-1}$ is an **environment** on base n;

  for some $n > 0$,
- or the (fresh) symbol $\Uparrow$ (**error**).

$\mathbb{C}$ denotes the set of all configurations.

- Intuition:

$$\left(\mathcal{T}, \vdash_* \mathbf{G}_0, \ldots, \vdash_* \mathbf{G}_{n-1}\right) \quad \approx \quad \dfrac{\begin{matrix} \vdots\,\pi \\ \vdash \mathbf{F}_0, \ldots, \mathbf{F}_{n-1} \end{matrix} \qquad \begin{matrix} \vdots \\ \vdash_* \mathbf{G}_0 \end{matrix} \quad \ldots \quad \begin{matrix} \vdots \\ \vdash_* \mathbf{G}_{n-1} \end{matrix}}{}\ \textbf{cut}$$

Reduction relation (I)

The **reduction relation** $\longrightarrow$ is the subset of $\mathbb{C} \times \mathbb{C}$ defined as follows.

(1) $\Uparrow \longrightarrow \Uparrow$.

Intuition: " error reduces to error."

Reduction relation (II)

(2) Let $C = \big(\langle n, i, a_0\rangle.\mathfrak{U}\,,\ \vdash_* \mathbf{G}_0\ \ \ldots\ \ \vdash_* \mathbf{G}_{n-1}\big)$.

- If $\mathbf{G}_i = \bigwedge_S \mathbf{F}_a$ and $a_0 \in S$, then
$$C \longrightarrow \big(\mathfrak{U}\,,\ \vdash_* \mathbf{G}_0\ \ \ldots\ \ \vdash_* \mathbf{G}_{n-1} \vdash_* \mathbf{F}_{a_0}\big).$$
- $C \longrightarrow \Uparrow$, otherwise.

Intuition (case $n = 2$ and $i = 1$):

$$\dfrac{\dfrac{\overset{\vdots\ \pi}{\vdash\ \mathbf{A}\,,\ \bigvee_T \mathbf{H}_a\,,\ \mathbf{H}_{a_0}}}{\vdash\ \mathbf{A}\,,\ \bigvee_T \mathbf{H}_a}\,(\vee) \qquad \overset{\vdots}{\vdash_* \mathbf{G}_0} \qquad \dfrac{\overset{\vdots}{\vdash_*\ \mathbf{F}_a} \quad \ldots \text{all } a \in S}{\vdash_* \bigwedge_S \mathbf{F}_a}}{\qquad}\ \textbf{cut}$$

reduces to

$$\dfrac{\overset{\vdots\ \pi}{\vdash\ \mathbf{A}\,,\ \bigvee_T \mathbf{H}_a\,,\ \mathbf{H}_{a_0}} \qquad \overset{\vdots}{\vdash_* \mathbf{G}_0} \qquad \dfrac{\overset{\vdots}{\vdash_*\ \mathbf{F}_a} \quad \ldots \text{all } a \in S}{\vdash_* \bigwedge_S \mathbf{F}_a} \qquad \overset{\vdots}{\vdash_* \mathbf{F}_{a_0}}}{\qquad}\ \textbf{cut}$$

Reduction relation (III)

(3) Let $C = \left([n,i].\mathfrak{U}_a \,,\, \vdash_* \mathbf{G}_0 \;\; \ldots \;\; \vdash_* \mathbf{G}_{n-1}\right)$.

- If $\mathbf{G}_i = \bigvee_S \mathbf{F}_a$, then

$$C \longrightarrow \left(\mathfrak{U}_a \,,\, \vdash_* \mathbf{G}_0 \;\; \ldots \;\; \vdash_* \mathbf{G}_{n-1} \vdash_* \mathbf{F}_a\right) \;\;,\;\; \text{for all } a \in S.$$

- $C \longrightarrow \Uparrow$, otherwise.

Intuition (case $n = 2$, $i = 1$):

$$\dfrac{\dfrac{\overset{\vdots\,\pi_a}{\vdash \mathbf{A}\,,\, \bigwedge_{\mathbb{N}} \mathbf{H}_a \,,\, \mathbf{H}_a \ldots \text{all } a \in \mathbb{N}}}{\vdash \mathbf{A}\,,\, \bigwedge_{\mathbb{N}} \mathbf{H}_a}\,(\wedge) \qquad \overset{\vdots}{\vdash_* \mathbf{G}_0} \qquad \dfrac{\overset{\vdots}{\vdash_* \;\; \mathbf{F}_a} \;\; \ldots \text{all } a \in S}{\vdash_* \bigvee_S \mathbf{F}_a}}{}\,\textbf{cut}$$

reduces to

$$\dfrac{\overset{\vdots\,\pi_a}{\vdash \mathbf{A}\,,\, \bigwedge_{\mathbb{N}} \mathbf{H}_a \,,\, \mathbf{H}_a} \qquad \overset{\vdots}{\vdash_* \mathbf{G}_0} \qquad \dfrac{\overset{\vdots}{\vdash_* \;\; \mathbf{F}_a} \;\; \ldots \text{all } a \in S}{\vdash_* \bigvee_S \mathbf{F}_a} \qquad \overset{\vdots}{\vdash_* \mathbf{F}_a}}{}\,\textbf{cut}$$

one cut for each $a \in S$.

Examples

- The configuration
$$\big([n,i].\mathfrak{U}_a\ ,\ \vdash_* \mathbf{G}_0\ \ldots\ \vdash_* \mathbf{G}_{i-1}\ \vdash_* \mathbf{0}\ \vdash_* \mathbf{G}_{i+1}\ \vdash_* \mathbf{G}_{n-1}\big)$$
does not reduce to anything (because $\mathbf{G}_i = \mathbf{0} = \bigvee_{\emptyset} \mathbf{F}_a$).

- The configuration
$$\big([1,0].\mathfrak{U}_a\ ,\ \vdash_* \bigvee\nolimits_{\{c,d\}} \mathbf{G}_a\big) \text{ reduces to}$$
$$(\mathfrak{U}_c\ ,\ \vdash_* \bigvee\nolimits_{\{c,d\}} \mathbf{G}_a \vdash_* \mathbf{G}_c) \text{ and } (\mathfrak{U}_d\ ,\ \vdash_* \bigvee\nolimits_{\{c,d\}} \mathbf{G}_a \vdash_* \mathbf{G}_d)$$
- Let $\mathfrak{T} \stackrel{\text{DEF}}{=} \langle 1,0,a_0\rangle.\langle 2,0,a_0\rangle \ldots \langle n,0,a_0\rangle.\langle n+1,0,a_0\rangle \ldots$ and $\mathbf{F} \stackrel{\text{DEF}}{=} \bigwedge_{\{a_0\}} \mathbf{G}_a$, where $\mathbf{G}_{a_0} \stackrel{\text{DEF}}{=} \mathbf{0}$. Then,

$$\begin{array}{lll}
(\mathfrak{T}\ ,\ \vdash_* \mathbf{F}) & \longrightarrow & (\langle 2,0,a_0\rangle \ldots\ ,\ \vdash_* \mathbf{F} \vdash_* \mathbf{0}) \\
 & \longrightarrow & \\
 & \vdots & \\
 & \longrightarrow & (\langle n,0,a_0\rangle.\langle n+1,0,a_0\rangle \ldots\ ,\ \vdash_* \mathbf{F} \vdash_* \mathbf{0} \ldots \vdash_* \mathbf{0}) \\
 & \longrightarrow & (\langle n+1,0,a_0\rangle \ldots\ ,\ \vdash_* \mathbf{F} \vdash_* \mathbf{0} \ldots \vdash_* \mathbf{0} \vdash_* \mathbf{0}) \\
 & \longrightarrow & \cdots
\end{array}$$

Some properties of $\longrightarrow$

Let A be a set and let R be a binary relation of A.

- R is **total** $\stackrel{\text{DEF}}{\Longleftrightarrow}$ for all $a \in A$ there is $b \in A$ such that $a \; R \; b$;
- R is **deterministic** $\stackrel{\text{DEF}}{\Longleftrightarrow}$ $a \; R \; b$ and $a \; R \; c$ imply $b = c$;
- R is **terminating** $\stackrel{\text{DEF}}{\Longleftrightarrow}$ there is no infinite sequence

$$a_0 \longrightarrow a_1 \longrightarrow \cdots.$$

The relation $\longrightarrow$ is **not total**,
not deterministic,
not terminating.

INT(Θ)

Let $\Theta = \vdash \mathbf{F}_0, \ldots, \mathbf{F}_{n-1}$ be a sequent of $\mathcal{T}$. We define the **interactive interpretation of Θ** as follows:

$\mathfrak{T} \in \mathbf{INT}(\Theta) \quad \overset{\text{DEF}}{\Longleftrightarrow}$ every sequence of reductions starting from $(\mathfrak{T}, \vdash_* \mathbf{F}_0^\perp \ldots \vdash_* \mathbf{F}_{n-1}^\perp)$ terminates.

Soundness–and–completeness

Application: additive connectives in ludics (I)

We say that a test $\mathfrak{T}$ is **affine** (or, improperly **linear**), if for every $s, t \in \mathtt{TREE}(\mathfrak{T})$ the following condition holds:

$$s \sqsubset t \quad \Longrightarrow \quad \text{the addresses of } \mathtt{LAB}(\mathfrak{T})(s) \text{ and } \mathtt{LAB}(\mathfrak{T})(t) \text{ are different.}$$

In the "formulas–as–resources" interpretation, this condition formalizes the idea that any occurrence of formula is used (decomposed) "**at most once**" in a **branch** of a derivation, i.e., only **additive contraction** (sharing of contexts) is allowed. In terms of rules:

$$\frac{\vdash \ \mathbf{F}_0, \ldots, \mathbf{F}_{i-1} \ , \quad \mathbf{0} \quad , \ \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{n-1} \ , \ \mathbf{G}_{a_0}}{\vdash \ \mathbf{F}_0, \ldots, \mathbf{F}_{i-1} \ , \ \bigvee_S \mathbf{G}_a \ , \ \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{n-1}} \ (\vee_{aff})$$

$$\frac{\vdash \ \mathbf{F}_0, \ldots, \mathbf{F}_{i-1} \ , \quad \mathbf{0} \quad , \ \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{n-1} \ , \ \mathbf{G}_a \qquad \ldots \text{all } a \in S}{\vdash \ \mathbf{F}_0, \ldots, \mathbf{F}_{i-1} \ , \ \bigwedge_S \mathbf{G}_a \ , \ \mathbf{F}_{i+1}, \ldots, \mathbf{F}_{n-1}} \ (\wedge_{aff})$$

We can use **0** to express the fact that "the slot i is unavailable."

Application: additive connectives in ludics (II)

Let $\mathbf{A} = \bigvee_S \mathbf{F}_a$, $\mathbf{B} = \bigvee_T \mathbf{F}_a$, $\mathbf{C} = \bigwedge_S \mathbf{G}_a$, $\mathbf{D} = \bigwedge_T \mathbf{G}_a$, and suppose that S and T are **disjoint.** Define:

$$\mathbf{A} \oplus \mathbf{B} \stackrel{\text{DEF}}{=} \bigvee_{S \cup T} \mathbf{F}_a;$$

$$\mathbf{C} \mathbin{\&} \mathbf{D} \stackrel{\text{DEF}}{=} \bigwedge_{S \cup T} \mathbf{G}_a.$$

$\mathfrak{T} \in \mathbf{INT}^\star(\mathbf{F}) \stackrel{\text{DEF}}{\Longleftrightarrow}$ $\mathfrak{T}$ **is affine**, and every sequence of reductions starting from $(\mathfrak{T} , \vdash_* \mathbf{F})$ terminates.

Then, one can show that:

$$\mathbf{INT}^\star(\mathbf{A} \oplus \mathbf{B}) = \mathbf{INT}^\star(\mathbf{A}) \cup \mathbf{INT}^\star(\mathbf{B});$$

$$\mathbf{INT}^\star(\mathbf{C} \mathbin{\&} \mathbf{D}) = \mathbf{INT}^\star(\mathbf{C}) \cap \mathbf{INT}^\star(\mathbf{D}).$$

Moreover, the union in the case of $\oplus$ is **disjoint.**

Soundness–and–completeness

For every sequent $\mathbf{\Theta}$ in $\mathcal{T}$:

$$\mathfrak{T} \in \mathbf{DER}(\mathbf{\Theta}) \iff \mathfrak{T} \in \mathbf{INT}(\mathbf{\Theta}).$$

Future work

- Propositional variables and **second order** quantifiers.
- Girard's β–logic (the logic underlying the theory of **dilators**).
- ...

Thank you!

Thank you!

Questions?

Thank you!

Questions?

Answers?