The Full Abstraction of Probabilistic Coherence Spaces

Thomas Ehrhard[*] Michele Pagani[+] Christine Tasson[*]

[+]Laboratoire d'Informatique de Paris Nord
Institut Galilée, Universitè de Paris Nord — Paris 13 (FR)
michele.pagani@lipn.univ-paris13.fr

[*]Laboratoire Preuves Programmes et Systèmes
Université Diderot — Paris 7 (FR)
{ehrhard, tasson}@pps.jussieu.fr

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Probabilistic Coherence Spaces

- introduced as a model of Multiplicative Additive Linear Logic

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- extended to full Linear Logic and λ-calculus

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$|\mathcal{A}|$ a set (possibly infinite), called *web*

$\mathrm{P}(\mathcal{A})$ a set of vectors $\subseteq (\mathbb{R}^+)^{|\mathcal{A}|}$ such that

closure: $\mathrm{P}(\mathcal{A})^{\perp\perp} = \mathrm{P}(\mathcal{A})$

- for all $v, u \in (\mathbb{R}^+)^{|\mathcal{A}|}$, let $\langle u, v \rangle = \sum_{a \in |\mathcal{A}|} u_a v_a$
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complete: $\forall a \in |\mathcal{A}|, \exists v \in \mathrm{P}(\mathcal{A}), v_a \neq 0$

bound: $\forall a \in |\mathcal{A}|, \exists p \in \mathbb{R}^+, \forall v \in \mathrm{P}(\mathcal{A}), v_a \leq p$

Example

$$|\mathrm{Bool}| = \{\mathtt{t}, \mathtt{f}\} \qquad |\mathrm{Nat}| = \{0, 1, 2, 3, \dots\}$$

$$\mathrm{P}(\mathrm{Bool}) = \{(p, q) \,;\, p + q \leq 1\} \qquad \mathrm{P}(\mathrm{Nat}) = \{v \in [0,1]^{\mathbb{N}} \,;\, \textstyle\sum_n v_n \leq 1\}$$

$$= \{(1,0), (0,1)\}^{\perp\perp} \qquad = \{\delta(n) \,;\, n \in \mathbb{N}\}^{\perp\perp}$$

$$|\mathrm{Bool} \Rightarrow \mathrm{Bool}| = \mathcal{M}_f(\{\mathtt{t}, \mathtt{f}\}) \times \{\mathtt{t}, \mathtt{f}\}$$

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Probabilistic Coherence Spaces $\quad \mathcal{A} = (|\mathcal{A}|, \mathrm{P}(\mathcal{A}))$

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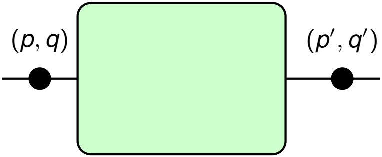
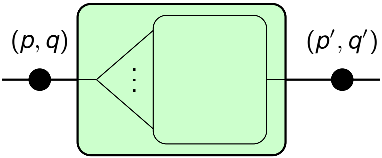


$(Px)_{\mathtt{t}} =$ **?**



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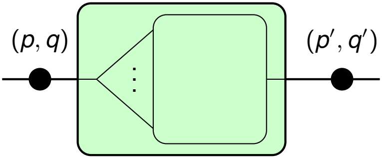
Modeling Programs on **Probabilistic** Data



$$
\begin{aligned}
(Px)_{\mathtt{t}} = & P_{[\,],\mathtt{t}} \\
& + P_{[\bullet],\mathtt{t}}\, x \\
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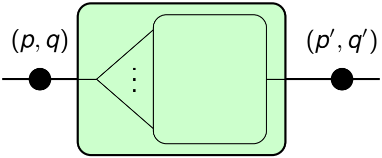

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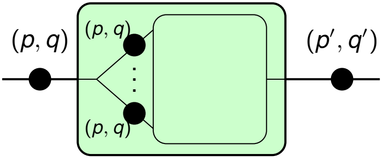

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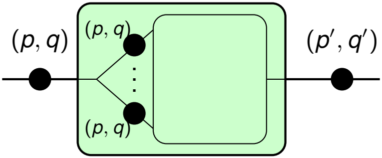

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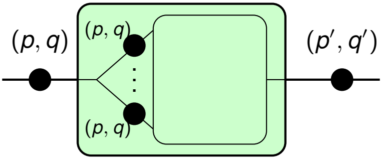

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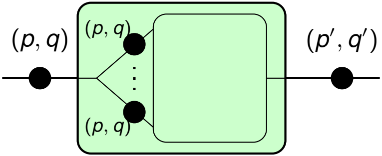

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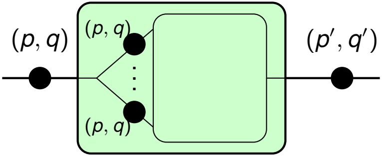

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&+ P_{[\mathtt{t},\mathtt{t},\mathtt{t}],\mathtt{t}} x_{\mathtt{t}}^3 + P_{[\mathtt{t},\mathtt{t},\mathtt{f}],\mathtt{t}} x_{\mathtt{t}}^2 x_{\mathtt{f}} + P_{[\mathtt{t},\mathtt{f},\mathtt{f}],\mathtt{t}} x_{\mathtt{t}} x_{\mathtt{f}}^2 + P_{[\mathtt{f},\mathtt{f},\mathtt{f}],\mathtt{t}} x_{\mathtt{f}}^3 \\
&\vdots \\
=& \sum_{m \in \mathcal{M}_{\mathrm{f}}(\{\mathtt{t},\mathtt{f}\})} P_{m,\mathtt{t}} x_{\mathtt{t}}^{m(\mathtt{t})} x_{\mathtt{f}}^{m(\mathtt{f})}
\end{aligned}
$$

$\Leftarrow$ power series in the unknowns $x_{\mathtt{t}}$ and $x_{\mathtt{f}}$.

objects: probabilistic coherence spaces

- $\mathcal{A} = (|\mathcal{A}|, \mathrm{P}(\mathcal{A}))$

morphisms: matrices $M \in \mathbb{R}^{+\mathcal{M}_f(|\mathcal{A}|)\times|\mathcal{B}|}$ such that $\forall x \in \mathrm{P}(\mathcal{A})$, $(Mx) \in \mathrm{P}(\mathcal{B})$,

- $(Mx)_b = \sum_{m \in \mathcal{M}_f(|\mathcal{A}|)} M_{m,b} \prod_{a \in \mathrm{Supp}(m)} x_a^{m(a)}$

Example

$$\mathbf{Id} = \begin{cases} [a], a & \mapsto 1 \\ \text{otherwise} & \mapsto 0 \end{cases}$$

$$\mathbf{Eval} = \begin{cases} ([(m,b)], m), b & \mapsto 1 \\ \text{otherwise} & \mapsto 0 \end{cases}$$

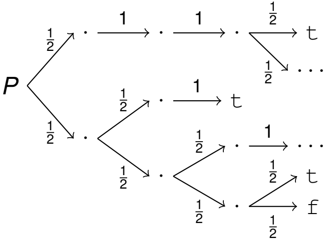
$$\mathbf{Random} = \begin{cases} [n], 0 & \mapsto \frac{1}{n} \\ [n], 1 & \mapsto \frac{1}{n} \\ \vdots & \\ [n], n-1 & \mapsto \frac{1}{n} \\ \text{otherwise} & \mapsto 0 \end{cases}$$

Probabilistic PCF

Types: Bool | Nat | $A \Rightarrow B$

Terms: | $\lambda x^A.P$ | $(P)\,Q$ | fix(P) | $\underline{0}$ | p(P) | s(P) | zero?(P) | `t` | `f` | if(N, P, Q) | `Coin`

Reduction: $P \xrightarrow{p} Q$ — P reduces to Q in one step with probability p



$$\texttt{Coin} \xrightarrow{\frac{1}{2}} \texttt{t} \qquad \texttt{Coin} \xrightarrow{\frac{1}{2}} \texttt{f}$$

$$\textbf{Prob}(P \xrightarrow{p_1} \dots \xrightarrow{p_k} Q) = \prod_{i=1}^{k} p_i$$

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Theorem (Danos, Ehrhard 2011)

For every closed term P of type Bool*:*

$$[\![P]\!]_{\texttt{t}} = \textbf{Prob}(P, \texttt{t}) \text{ and } [\![P]\!]_{\texttt{f}} = \textbf{Prob}(P, \texttt{f})$$

and the same for Nat.

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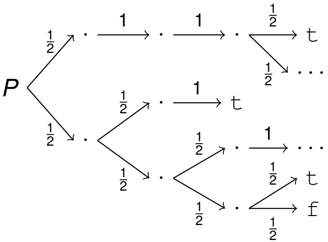

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Observational Equivalence

Definition

For $\Gamma \vdash P : A$ and $\Gamma \vdash Q : A$, define:

$$P \sim Q \quad \text{iff} \quad \forall C \text{ context}, \; \mathbf{Prob}(C[P], \mathtt{t}) = \mathbf{Prob}(C[Q], \mathtt{t})$$

Theorem (Ehrhard-Pagani-Tasson 2013)

For $\Gamma \vdash P : A$ and $\Gamma \vdash Q : A$,

$$[\![P]\!] = [\![Q]\!] \quad \text{iff} \quad P \sim Q$$

Proof.

$\Longrightarrow$: immediate consequence of the adequacy theorem

$\Longleftarrow$: our work !

- suppose $[\![P]\!] \neq [\![Q]\!]$,
- find a context C, such that

$$\mathbf{Prob}(C[P], \mathtt{t}) \neq \mathbf{Prob}(C[Q], \mathtt{t})$$

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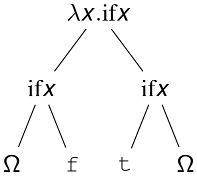
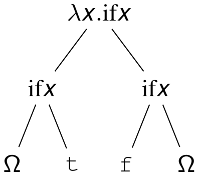


$\sim$



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One powerful context in `Bool`: $\_\_\ p\mathtt{t} \oplus q\mathtt{f}$

$\mathbf{Prob}(C[\Omega], \mathtt{t}) = 0$ $\mathbf{Prob}(C[P], \mathtt{t}) = qp$ $\mathbf{Prob}(C[Q], \mathtt{t}) = pq$

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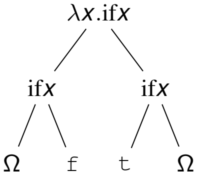
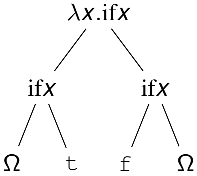
$\lambda x.\Omega$ $\not\sim$ 

$\sim$ 

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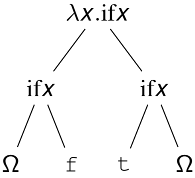

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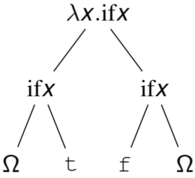

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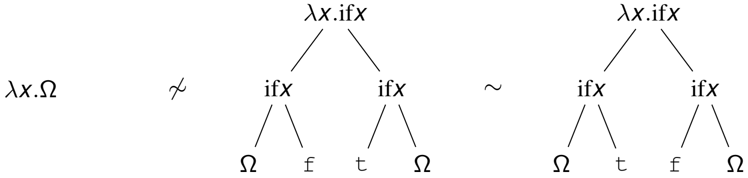

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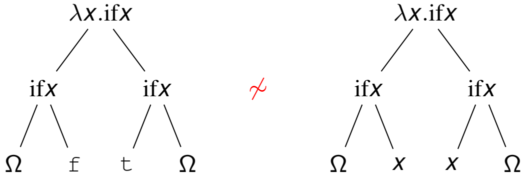

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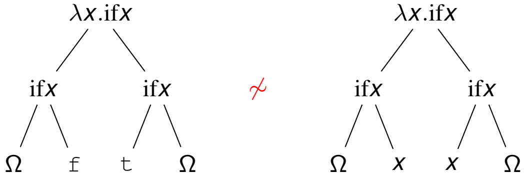

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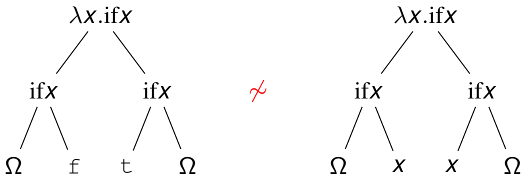

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On Generic Types $B \Rightarrow C$

in general, power series are much more complex:

$$\sum_{m \in \mathcal{M}_f(|\mathcal{B}|)} [\![P]\!]_{m,c} \prod_{b \in \mathrm{Supp}(m)} x_b^{m(b)}$$

- ✘ $(m, c) \in |\mathcal{B} \Rightarrow \mathcal{C}|$ might not correspond to a term in the syntax.
- ✘ the number of unknowns $x_{(m,c)}$ might be infinite.

Our solution

$\forall a \in |\mathcal{A}|$, define $F^a : A \Rightarrow \mathtt{Bool}^k \Rightarrow \mathtt{Bool}$, such that:

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A glance at the definition of F^a

Given $a \in |\mathcal{A}|$, we define by mutual induction two terms:

$$\texttt{Bool}^{k^a} \vdash F^a : A \Rightarrow \texttt{Bool}$$
$$\texttt{Bool}^{k^a} \vdash N^a : A$$

Definition

if $A = \texttt{Bool}, \texttt{Nat}$,

$$F^b = \lambda x^\iota.\text{if}(x = b, \texttt{t}, \Omega)$$
$$N^b = b$$

if $A = C \Rightarrow D$,

$$F^{([c_1,\dots,c_h],d)} = \lambda x^{C \Rightarrow D}.\left(F^d\right)\left((x)\bigoplus_{i=1}^{h}(z_i \cdot N^{c_i})\right)$$
$$N^{([c_1,\dots,c_h],d)} = \lambda x^C.\text{if}(\wedge_{i=1}^{h}\left(F^{c_i}\right) x, N^d, \Omega)$$

A glance at the weighted intersection type system

$$\frac{}{x^A : [a] \vdash_1 x : a} \qquad \frac{}{\vdash_1 \underline{n} : n} \qquad \frac{b \in \{\mathtt{t}, \mathtt{f}\}}{\vdash_{\frac{1}{2}} \mathtt{Coin} : b} \qquad \frac{\Gamma^\bullet, x^A : m \vdash_\alpha M : a}{\Gamma^\bullet \vdash_\alpha \lambda x^A.M : (m, a)}$$

$$\frac{\Gamma^{\bullet\prime} \vdash_\alpha M : (m, b) \qquad \forall (a,i) \in m, \quad \Gamma^\bullet_{(a,i)} \vdash_{\beta_{(a,i)}} N : a_{(a,i)}}{\Gamma^{\bullet\prime} \uplus \biguplus_{(a,i)\in m} \Gamma^\bullet_{(a,i)} \vdash_{\alpha \prod_{(a,i)\in m} \beta_{(a,i)}} (M)\, N : b}$$

$$\frac{\Gamma^{\bullet\prime} \vdash_\alpha M : (m, b) \qquad \forall (a,i) \in m, \quad \Gamma^\bullet_{(a,i)} \vdash_{\beta_{(a,i)}} \mathsf{fix}(M) : a_{(a,i)}}{\Gamma^{\bullet\prime} \uplus \biguplus_{(a,i)\in m} \Gamma^\bullet_{(a,i)} \vdash_{\alpha \prod_{(a,i)\in m} \beta_{(a,i)}} \mathsf{fix}(M) : b}$$

$$\frac{\Gamma^\bullet \vdash_\alpha M : n+1}{\Gamma^\bullet \vdash_\alpha \mathsf{p}(M) : n}\ \text{pred} \qquad \frac{\Gamma^\bullet \vdash_\alpha M : n}{\Gamma^\bullet \vdash_\alpha \mathsf{s}(M) : n+1} \qquad \frac{\Gamma^\bullet \vdash_\alpha M : a}{\Gamma^\bullet \vdash_{\alpha X} X \cdot M : a}$$

$$\frac{\Gamma^\bullet \vdash_\beta M : 0 \qquad \Delta^\bullet \vdash_\alpha N : a}{\Gamma^\bullet \uplus \Delta^\bullet \vdash_{\beta\alpha} \mathsf{if}(M, N, P) : a} \qquad \frac{\Gamma^\bullet \vdash_\beta M : n+1 \qquad \Delta^\bullet \vdash_\alpha P : a}{\Gamma^\bullet \uplus \Delta^\bullet \vdash_{\beta\alpha} \mathsf{if}(M, N, P) : a}$$