Toward a New Theory of Exponential Proof Nets

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MELL

- **Proof nets**: the **graphical syntax** for linear logic.
- **MELL**: linear logic **without the additives**.
- Brought new deep perspectives about **normalization**:
  1. Optimal reductions;
  2. Geometry of interaction;
  3. Implicit computational complexity;
  4. Explicit substitutions;
- **Key concept**: **boxes** for the promotion rule, the heart of the system.

Problems

- **Statics**:
  1. **Boxes**: can they be *induced* by some logic feature?
  2. **Correctness**: no correctness criteria!
- **Dynamics**:
  1. **Strong Normalization**: shouldn't it be easy?
  2. **Confluence**: no residuals/cube property.
  3. **Standardization**: folklore theory.

Toward a New Theory of Exponential Proof Nets

- **Statics**:

  1. **Boxes**: can they be *induced* by some logic feature?
     This talk, *LICS 2013*.
  2. **Correctness**: no correctness criteria!
     Conjecture: Intuitionistic MELL has a criterion, MELL does not.

- **Dynamics**:

  1. **Strong Normalization**: shouldn't it be easy?
     This talk, *RTA 2013*.
  2. **Confluence**: no residuals/cube property.
     A-Bonelli-Kesner-Lombardi, *POPL 2014*.
  3. **Standardization**: folklore theory.
     A-Bonelli-Kesner-Lombardi, *POPL 2014*.

Compressing Polarized Boxes

Multiplicative Linear Logic (MLL)

Identity rules:

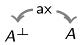
$$\frac{}{\vdash A^{\perp}, A}\,\mathsf{ax} \qquad \frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\,\mathsf{cut}$$

Multiplicative rules:

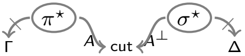
$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\,\otimes \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr$$

Proof nets for MLL

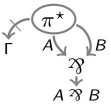
$$\frac{}{\vdash A^{\perp}, A}\ \mathsf{ax} \quad \rightsquigarrow$$



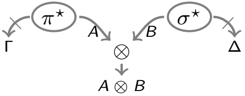
$$\frac{\begin{array}{c}\pi\\ \vdots\\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\sigma\\ \vdots\\ \vdash \Delta, A^{\perp}\end{array}}{\vdash \Gamma, \Delta}\ \mathsf{cut} \quad \rightsquigarrow$$



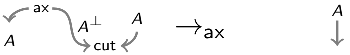
$$\frac{\begin{array}{c}\pi\\ \vdots\\ \vdash \Gamma, A, B\end{array}}{\vdash \Gamma, A \parr B}\ \parr \quad \rightsquigarrow$$

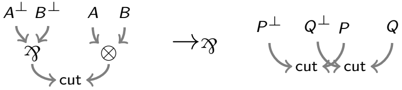


$$\frac{\begin{array}{c}\pi\\ \vdots\\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\sigma\\ \vdots\\ \vdash \Delta, B\end{array}}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes \quad \rightsquigarrow$$



Cut-elimination for MLL





No duplication/erasure of subnets

$\Rightarrow$

Everything works **fine**

Multiplicative Exponential Linear Logic (MELL)

MLL

$+$

Exponential rules:

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\,\mathsf{d} \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,!$$

$$\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\,\mathsf{c} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\,\mathsf{w}$$

Exponential Cut-elimination

Consider the following cut with **contraction**:

$$\dfrac{\dfrac{\begin{array}{c}\rho\\ \vdots\\ \vdash ?\Delta, A\end{array}}{\vdash ?\Delta, !A}\,! \qquad \dfrac{\begin{array}{c}\pi\\ \vdots\\ \vdash ?A^{\perp}, ?A^{\perp}, \Gamma\end{array}}{\vdash ?A^{\perp}, \Gamma}\,\mathsf{c}}{\vdash ?\Delta, \Gamma}\,\mathsf{cut}$$

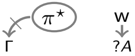
Its elimination requires to **duplicate** ρ:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\rho\\ \vdots\\ \vdash ?\Delta, A\end{array}}{\vdash ?\Delta, !A}\,! \qquad \dfrac{\dfrac{\begin{array}{c}\rho\\ \vdots\\ \vdash ?\Delta, A\end{array}}{\vdash ?\Delta, !A}\,! \qquad \begin{array}{c}\pi\\ \vdots\\ \vdash ?A^{\perp}, ?A^{\perp}, \Gamma\end{array}}{\vdash ?\Delta, ?A^{\perp}, \Gamma}\,\mathsf{cut}}{\dfrac{\vdash ?\Delta, ?\Delta, \Gamma}{\vdots}\,\mathsf{c}}\,\mathsf{cut}}{\vdash ?\Delta, \Gamma}\,\mathsf{c}$$

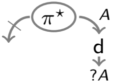
Similarly, **weakening** induces **erasure** of sub-proofs.

Naïve proof nets for MELL

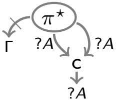
$$\dfrac{\begin{array}{c}\pi \\ \vdots \\ \vdash \Gamma\end{array}}{\vdash \Gamma, ?A}\,\mathsf{w} \quad \rightsquigarrow$$



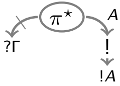
$$\dfrac{\begin{array}{c}\pi \\ \vdots \\ \vdash \Gamma, A\end{array}}{\vdash \Gamma, ?A}\,\mathsf{d} \quad \rightsquigarrow$$



$$\dfrac{\begin{array}{c}\pi \\ \vdots \\ \vdash \Gamma, ?A, ?A\end{array}}{\vdash \Gamma, ?A}\,\mathsf{c} \quad \rightsquigarrow$$

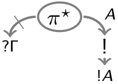


$$\dfrac{\begin{array}{c}\pi \\ \vdots \\ \vdash ?\Gamma, A\end{array}}{\vdash ?\Gamma, !A}\,! \quad \rightsquigarrow$$

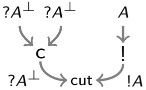


Naïve translation of promotion:

$$\dfrac{\begin{array}{c}\pi\\ \vdots\\ \vdash ?\Gamma, A\end{array}}{\vdash ?\Gamma, !A}\;! \qquad \rightsquigarrow$$



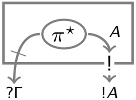
Given this cut in a generic net:



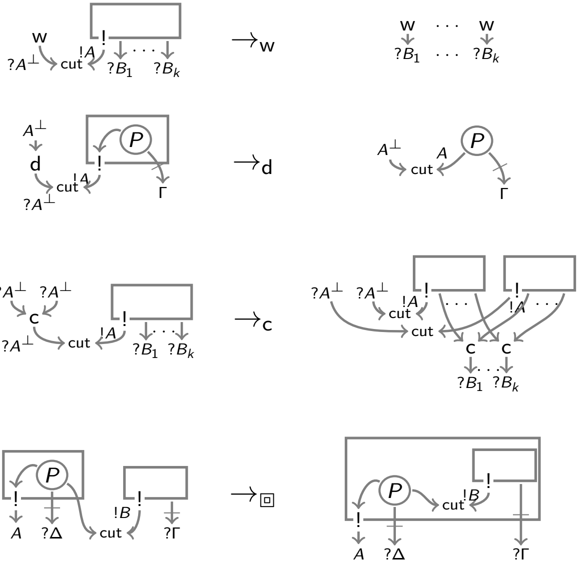
There is **no way** of recovering the **sub-proof to duplicate**.

Then **!-rules** are represented as **boxes**:

$$\dfrac{\begin{array}{c}\pi\\ \vdots\\ \vdash ?\Gamma, A\end{array}}{\vdash ?\Gamma, !A}\;! \qquad \rightsquigarrow$$



Exponential cut elimination implemented using boxes



Boxes

- Boxes **solve the problem** of defining cut-elimination.
- However, the solution is **drastic**, equivalent to **give up**.
- Some fragments seem to have an **inherent notion of box**.
- Where does the problem lie?
- Is there a **logic feature** that **internalizes boxes**?

Last rule 1

- **Main problem**: in proof nets there is **no last rule**.
- Re-consider:

$$
\dfrac{\dfrac{\begin{matrix}\rho\\ \vdots\\ \vdash ?\Delta, A\end{matrix}}{\vdash ?\Delta, !A}\,!
\qquad
\dfrac{\begin{matrix}\pi\\ \vdots\\ \vdash ?A^{\perp}, ?A^{\perp}, \Gamma\end{matrix}}{\vdash ?A^{\perp}, \Gamma}\,\mathsf{c}}{\vdash ?\Delta, \Gamma}\,\mathsf{cut}
\quad\longrightarrow\quad
\dfrac{\dfrac{\vdots}{\dfrac{\dfrac{\begin{matrix}\rho\\ \vdots\\ \vdash ?\Delta, A\end{matrix}}{\vdash ?\Delta, !A}\,!
\qquad
\dfrac{\dfrac{\begin{matrix}\rho\\ \vdots\\ \vdash ?\Delta, A\end{matrix}}{\vdash ?\Delta, !A}\,!
\qquad
\begin{matrix}\pi\\ \vdots\\ \vdash ?A^{\perp}, ?A^{\perp}, \Gamma\end{matrix}}{\vdash ?\Delta, ?A^{\perp}, \Gamma}\,\mathsf{cut}}{\vdash ?\Delta, ?\Delta, \Gamma}\,\mathsf{cut}}\,\mathsf{c}}{\vdash ?\Delta, \Gamma}\,\mathsf{c}
$$

- In sequent calculus:

rule occurrence $r \mapsto$ **sub-proof** ending on r.

- **No such thing in proof nets**!

Last rule 2

- **Intuition**:

  Internalizing a notion of **last rule**

  will internalize **boxes**

- **Partially internalized boxes**: Olivier Laurent's **MELLP**.
- **Abstract last rule** = **last positive rule**.
- **Expressiveness**: MELLP codes **classical logic**/$\lambda\mu$-calculus.
- **My contribution**: **totally internalized boxes** for MELLP.

Outline

Polarization

Formulas:

$$\begin{array}{llllllllll} P, Q ::= & X & \mid & 1 & \mid & P \otimes Q & \mid & !N \\ N, M ::= & X^{\perp} & \mid & \perp & \mid & N \parr M & \mid & ?P \end{array}$$

Sequents:

$$\vdash \Gamma ; P \quad \text{or} \quad \vdash \Gamma ; \_$$

Multiplicative rules:

$$\frac{\vdash \Gamma; P \qquad \vdash \Delta, P^{\perp}; [Q]}{\vdash \Gamma, \Delta; [Q]}\ \text{cut} \qquad \frac{}{\vdash P^{\perp}; P}\ \text{ax}$$

$$\frac{\vdash \Gamma; [P]}{\vdash \Gamma, \perp; [P]}\ \perp \qquad \frac{}{\vdash; 1}\ 1$$

$$\frac{\vdash \Gamma, N, M; [P]}{\vdash \Gamma, N \parr M; [P]}\ \parr \qquad \frac{\vdash \Gamma; P \qquad \vdash \Delta; Q}{\vdash \Gamma, \Delta; P \otimes Q}\ \otimes$$

Laurent's MELLP: adding exponentials

- Exponential rules:

$$\frac{\vdash \Gamma; [P]}{\vdash \Gamma, N; [P]}\ \mathsf{w} \qquad \frac{\vdash \Gamma; P}{\vdash \Gamma, ?P; \_}\ \mathsf{d}$$

$$\frac{\vdash \Gamma, N, N; [P]}{\vdash \Gamma, N; [P]}\ \mathsf{c} \qquad \frac{\vdash \Gamma, N; \_}{\vdash \Gamma; !N}\ !$$

- **Difference with linear logic**:

  Promotion, contraction, and weakening **do not need** the ? modality.

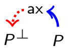
- **Important**:

  Only **positives** are duplicated/erased.

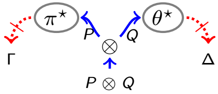
- Positives are last rules and **every positive** will have a **box**.

$$\frac{}{\vdash; 1}\;1 \qquad \leadsto$$

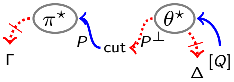
$$\frac{}{\vdash P^{\perp}; P}\;\mathsf{ax} \qquad \leadsto$$



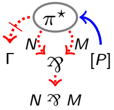
$$\frac{\vdash \Gamma; P \qquad \vdash \Delta; Q}{\vdash \Gamma, \Delta; P \otimes Q}\;\otimes \qquad \leadsto$$



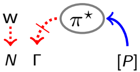
$$\frac{\vdash \Gamma; P \qquad \vdash \Delta, P^{\perp}; [Q]}{\vdash \Gamma, \Delta; [Q]}\;\mathsf{cut} \qquad \leadsto$$



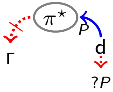
$$\frac{\vdash \Gamma, N, M; [P]}{\vdash \Gamma, N \parr M; [P]}\;\parr \qquad \leadsto$$



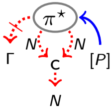
$$\frac{\vdash \Gamma; [P]}{\vdash \Gamma, N; [P]}\ \mathsf{w} \quad \rightsquigarrow$$



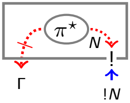
$$\frac{\vdash \Gamma; P}{\vdash \Gamma, ?P; \_}\ \mathsf{d} \quad \rightsquigarrow$$



$$\frac{\vdash \Gamma, N, N; [P]}{\vdash \Gamma, N; [P]}\ \mathsf{c} \quad \rightsquigarrow$$



$$\frac{\vdash \Gamma, N; \_}{\vdash \Gamma; !N}\ ! \quad \rightsquigarrow$$



Positive Trees

$$\frac{\vdash \Gamma; P \qquad \vdash \Delta, P^{\perp}; [Q]}{\vdash \Gamma, \Delta; [Q]}\ \mathsf{cut} \qquad \frac{}{\vdash P^{\perp}; P}\ \mathsf{ax}$$

$$\frac{\vdash \Gamma; [P]}{\vdash \Gamma, \perp; [P]}\ \perp \qquad \frac{}{\vdash; 1}\ 1$$

$$\frac{\vdash \Gamma, N, M; [P]}{\vdash \Gamma, N \parr M; [P]}\ \parr \qquad \frac{\vdash \Gamma; P \qquad \vdash \Delta; Q}{\vdash \Gamma, \Delta; P \otimes Q}\ \otimes$$

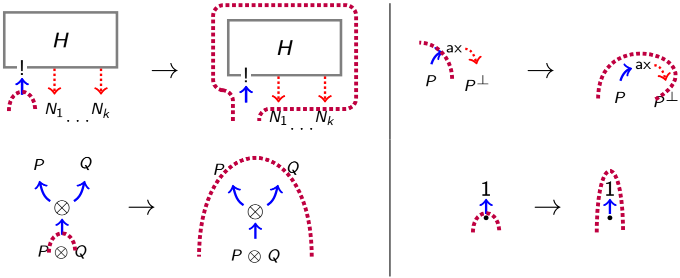
$$\frac{\vdash \Gamma; [P]}{\vdash \Gamma, N; [P]}\ \mathsf{w} \qquad \frac{\vdash \Gamma; P}{\vdash \Gamma, ?P; \_}\ \mathsf{d}$$

$$\frac{\vdash \Gamma, N, N; [P]}{\vdash \Gamma, N; [P]}\ \mathsf{c} \qquad \frac{\vdash \Gamma, N; \_}{\vdash \Gamma; !N}\ !$$

Note: positives have a **forest structure**.

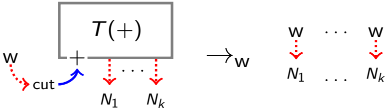
Positive Tree

- **Positive** connectives: $1, \otimes, !$.
- **Explicit** boxes for $!$ $\Rightarrow$ **induced box** for every positive:

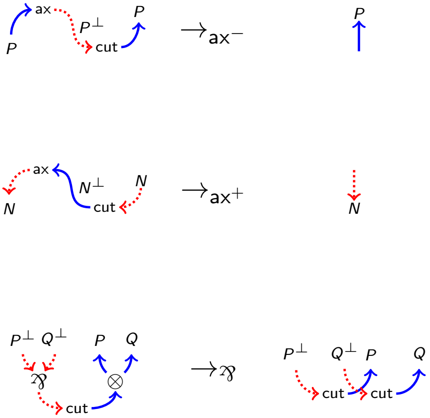


Generalized rewriting rule

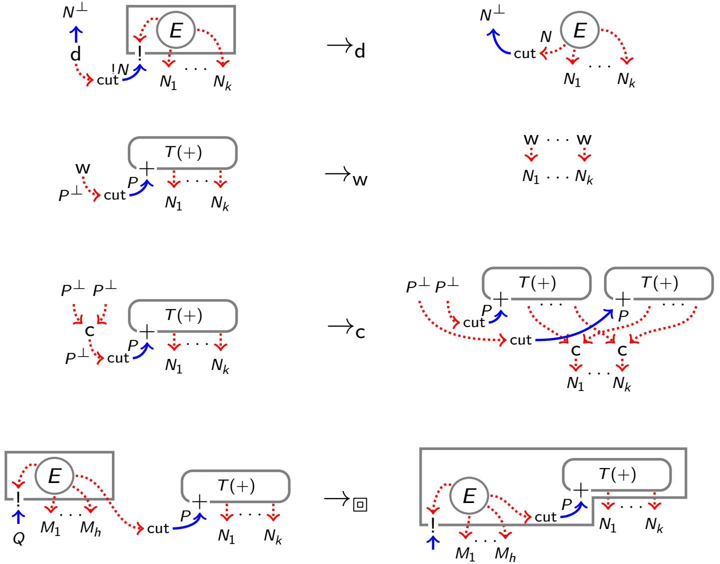
Laurent uses the **positive tree** to generalize box rules:



Polarized cut-elimination 1



Polarized cut-elimination 2



Outline

Matching property

$$\dfrac{\vdash \Gamma; P \qquad \vdash \Delta, P^{\perp}; [Q]}{\vdash \Gamma, \Delta; [Q]}\ \text{cut} \qquad \dfrac{}{\vdash P^{\perp}; P}\ \text{ax}$$

$$\dfrac{\vdash \Gamma; [P]}{\vdash \Gamma, \perp; [P]}\ \perp \qquad \dfrac{}{\vdash ; 1}\ 1$$

$$\dfrac{\vdash \Gamma, N, M; [P]}{\vdash \Gamma, N \parr M; [P]}\ \parr \qquad \dfrac{\vdash \Gamma; P \qquad \vdash \Delta; Q}{\vdash \Gamma, \Delta; P \otimes Q}\ \otimes$$

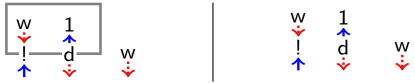
$$\dfrac{\vdash \Gamma; [P]}{\vdash \Gamma, N; [P]}\ \text{w} \qquad \dfrac{\vdash \Gamma; P}{\vdash \Gamma, ?P; \_}\ \text{d}$$

$$\dfrac{\vdash \Gamma, N, N; [P]}{\vdash \Gamma, N; [P]}\ \text{c} \qquad \dfrac{\vdash \Gamma, N; \_}{\vdash \Gamma; !N}\ !$$

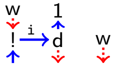
Matching property: every !-rule is **enabled** by a d-rule.

Materializing the matching property 1

- Consider:



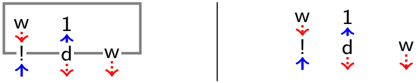
- **Problem**: without box the content is disconnected.
- **Idea**: let's materialize the matching property with an additional edge.



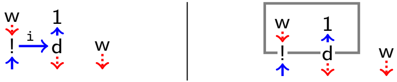
- The content and the positive sub-graphs are now **connected**.
- The **implicit box** of a !-link is its **matching dereliction**.
- The **induced box** is the **positive tree** plus the **negative trees** on it.

Quotient and weakenings

- Let's do it again:



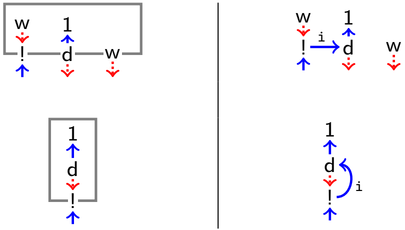
- We do not recover the original box:



- **Interpretation**: we are **quotienting** proof nets with explicit boxes.
- **Remark**: weakenings are not attached!
- $\Rightarrow$ improvement over **Francois Lamarche's essential nets**.

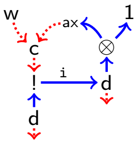
Box borders

- Let's do it again:



- **Remark**: we are not attaching the border of the box.
- $\Rightarrow$ improvement over **Ian Mackie's interaction nets** technique.

Correctness Criterion



Correctness Criterion: **Laurent's + Lamarche's**.

The conditions have to be **smoothed**:

- **Root**: exactly one initial node (no change).
- **Acyclicity**: every directed cycle uses the **negative premise** of a !.
- **Box condition**: If $p \rightarrow^{*} !$ then $! \rightarrow^{*} p$.

Implicit boxes

Recipe:

- Take a **cut-free proof net**.
- **Matching**: every !-box has a **unique dereliction at level 0**.
- **Remove the explicit box** and **add the matching edge**.

Then:

- The **induced boxes** define a net with explicit boxes.
- Induced boxes are **locally reconstructable**.
- There is a simple **correctness criterion** (i.e. not *ad-hoc*).
- It is a **canonical** representation (i.e. no choice).

Summing up

In a **cut-free** proof net

the **explicit box** of a !

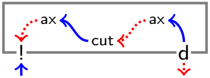
can be replaced by a **single edge**

in a **canonical** and **more parallel** way.

Outline

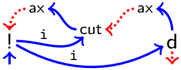
Cuts

Cuts introduce a **problem**:



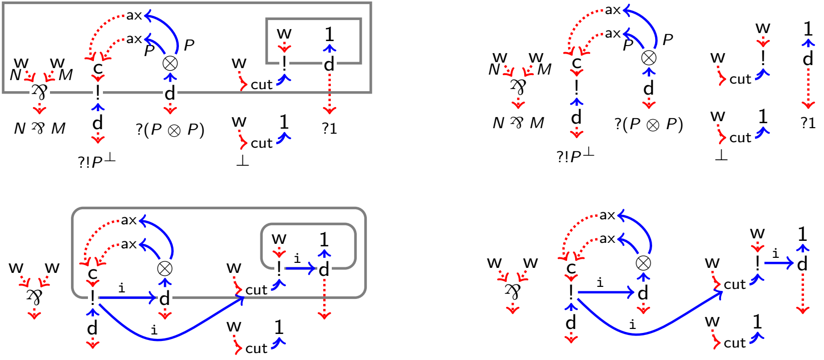
The **positive sub-graph** is **no longer connected**.

Let's iterate **the same idea**:



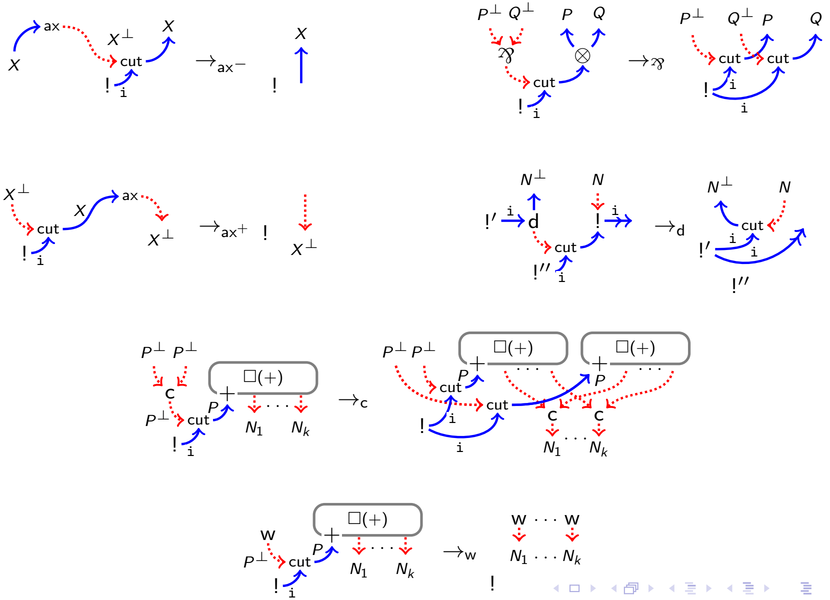
Implicit box of a !: matching dereliction + **cuts at level 0**.

Example



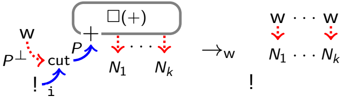
- **Induced box**: **positive tree** plus **negative sub-trees**.
- **Novelty**: ⅋ commutes with box borders!

Cut elimination

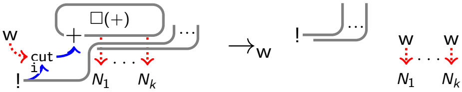


Side effects 1

- Cut elimination has '**side effects**'.
- Consider the weakening rule:



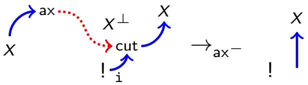
- It **automatically pushes the created weakenings out of boxes**:



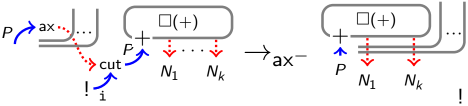
- Similarly for **contraction**.

No commutative cuts

- There is **no commutative rule**.
- It is included inside the axiom and dereliction rules.
- The axiom rule:

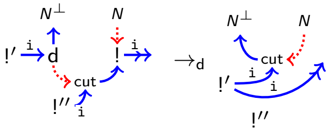


and its **action through box borders**:

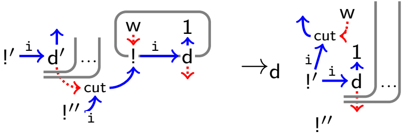


No commutative cuts 2

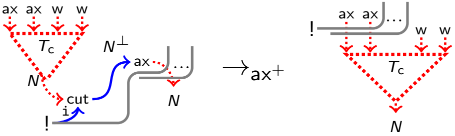
The dereliction rule:



and an example of its action:

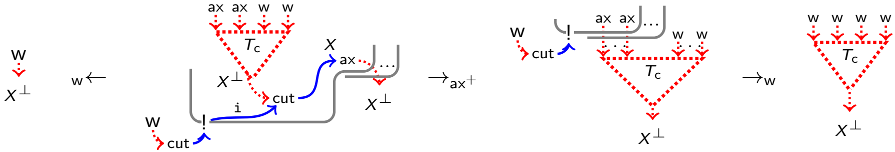


Side effects 2



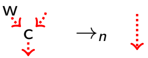
Axiom cuts make **whole negative trees** commute with box borders.

Critical pair:

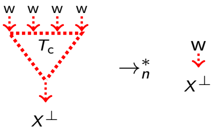


Neutrality

Neutrality of weakenings has to be added:



And how it solves the critical pair:



A similar example with contraction requires:

Very natural: these are rules of a **(co)commutative (co)monoid**.

Cut-elimination

On η-**expanded nets**:

- Cut elimination is **strongly normalizing** (SN).
- It is also **locally confluent**,
- and it **behaves nicely wrt ?-comonoids**.
- Then: **Church-Rosser modulo** and **SN modulo** hold.

Proof of SN

- **Algebraic** MELL(**P**) = MELL(**P**) + **?-comonoid** +

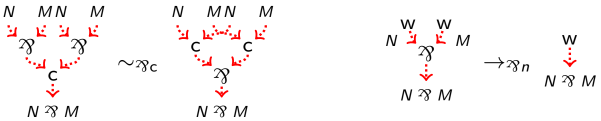
- Laurent's mapping MELLP $\hookrightarrow$ MELL lifts.
- Simulation:

  η**-exp. implicit** MELL**P** $\hookrightarrow$ **algebraic** MELL**P** $\hookrightarrow$ **algebraic** MELL

- **Pagani & Tranquilli**: algebraic MELL is SN.
- Then, η**-expanded implicit PN are SN**.

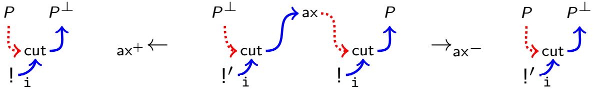
Side effects 3

- The **general case**: more and wilder '**side effects**'
- Two additional algebraic laws (for ⅋-comonoids) are required:



Problem: Laurent's translation does not validate them.

- And a new **problematic critical pair**:



Problem: To close the pair you need to look at the context.

The general case

- The new laws require to prove **SN from scratch**.
- The new critical pair makes **local confluence** challenging.
- **Damn!** All known proofs of SN are based on local confluence!
- **Natural question**: is local confluence really necessary for SN?

Strong Normalization

History

- **Girard, TCS '87**: linear logic (LL) and strong normalization (SN).
- A crucial lemma about the exponentials was left **unproven**.
- **Danos, PhD '90**: elaborated proof for second order MELL.
- Various other people worked on **SN for LL**:
  Joinet, van Raamsdonk, Okada, Di Cosmo & Guerrini.
- **Tortora de Falco and Pagani, TCS '10**: SN for second order LL.
- **Complex and long proof**, requiring confluence.
- **Here**: a simple and understandable proof, no need for confluence.

Outline

Kinds of cut

There are **two kinds of cut-elimination cases**.

1) **Principal**, *i.e.* the last rules introduce the cut formulas:

$$\dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\\ \vdash ?\Gamma, A\end{matrix}}{\vdash ?\Gamma, !A}\,! \qquad \dfrac{\begin{matrix}\theta\\ \vdots\\ \vdash \Delta, A^{\perp}\end{matrix}}{\vdash \Delta, ?A^{\perp}}\,\mathsf{d}}{\vdash ?\Gamma, \Delta}\,\mathsf{cut} \quad \to \quad \dfrac{\begin{matrix}\pi\\ \vdots\\ \vdash ?\Gamma, A\end{matrix} \qquad \begin{matrix}\theta\\ \vdots\\ \vdash \Delta, A^{\perp}\end{matrix}}{\vdash \Gamma, \Delta}\,\mathsf{cut}$$

2) **Commutative**, one last rule has no relation with the cut formula:

$$\dfrac{\begin{matrix}\pi\\ \vdots\\ \vdash ?\Gamma, !A\end{matrix} \qquad \dfrac{\begin{matrix}\theta\\ \vdots\\ \vdash ?A^{\perp}, ?\Delta, B\end{matrix}}{\vdash ?A^{\perp}, ?\Delta, !B}\,!}{\vdash ?\Gamma, ?\Delta, !B}\,\mathsf{cut} \quad \to \quad \dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\\ \vdash ?\Gamma, !A\end{matrix} \qquad \begin{matrix}\theta\\ \vdots\\ \vdash ?A^{\perp}, ?\Delta, B\end{matrix}}{\vdash ?\Gamma, ?\Delta, B}\,\mathsf{cut}}{\vdash ?A^{\perp}, ?\Delta, !B}\,!$$

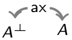
Commutative cases

- Commutative cases are **the burden** of cut-elimination.
- **Problem**: the cut rule commutes with itself.
- **Consequence**: silly diverging reductions.
- **Solution**:
  Switch to proof nets, where commutative cases (mostly) disappear.

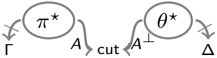
From sequent calculus to proof nets

The **multiplicative fragment**:

$$\frac{}{\vdash A^{\perp}, A}\ \mathsf{ax} \quad \rightsquigarrow$$



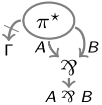
$$\frac{\begin{array}{c}\pi\\ \vdots\\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\sigma\\ \vdots\\ \vdash \Delta, A^{\perp}\end{array}}{\vdash \Gamma, \Delta}\ \mathsf{cut} \quad \rightsquigarrow$$



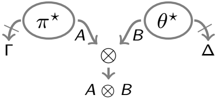
$$\frac{}{\vdash}\ \mathsf{mix}_0 \quad \rightsquigarrow$$

$$\frac{\begin{array}{c}\pi\\ \vdots\\ \vdash \Gamma\end{array} \qquad \begin{array}{c}\sigma\\ \vdots\\ \vdash \Delta\end{array}}{\vdash \Gamma, \Delta}\ \mathsf{mix}_2 \quad \rightsquigarrow$$

$$\frac{\begin{array}{c}\pi\\ \vdots\\ \vdash \Gamma, A, B\end{array}}{\vdash \Gamma, A \parr B}\ \parr \quad \rightsquigarrow$$

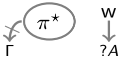


$$\frac{\begin{array}{c}\pi\\ \vdots\\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\sigma\\ \vdots\\ \vdash \Delta, B\end{array}}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes \quad \rightsquigarrow$$

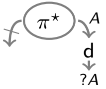


The **exponential fragment**:

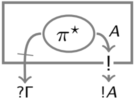
$$\dfrac{\begin{array}{c}\pi\\ \vdots\\ \vdash \Gamma\end{array}}{\vdash \Gamma, ?A}\,\mathsf{w} \quad\rightsquigarrow$$



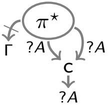
$$\dfrac{\begin{array}{c}\pi\\ \vdots\\ \vdash \Gamma, A\end{array}}{\vdash \Gamma, ?A}\,\mathsf{d} \quad\rightsquigarrow$$



$$\dfrac{\begin{array}{c}\pi\\ \vdots\\ \vdash ?\Gamma, A\end{array}}{\vdash ?\Gamma, !A}\,! \quad\rightsquigarrow$$



$$\dfrac{\begin{array}{c}\pi\\ \vdots\\ \vdash \Gamma, ?A, ?A\end{array}}{\vdash \Gamma, ?A}\,\mathsf{c} \quad\rightsquigarrow$$



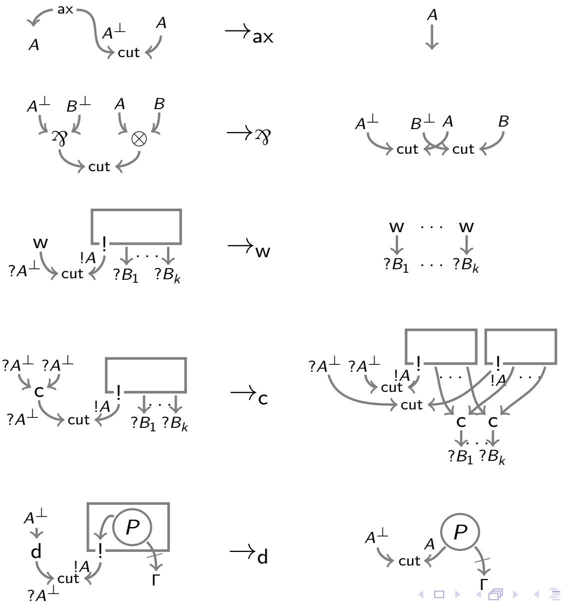
Black-box principle

- Girard introduced boxes according to the **black-box principle**:

  > *"boxes are treated in a perfectly* **modular way***: we can use the box B without knowing its contents, i.e., another box B' with exactly the same doors would do as well"*

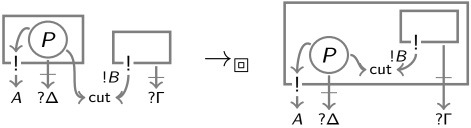
- **Principal cases**: 2 deductive rules cut at level 0 in the same box.
- **Only one commutative case**:
  a rule moving boxes to bring premises of a cut at the same box level

Proof nets cut-elimination: principal cases



Proof nets cut-elimination: the commutative case

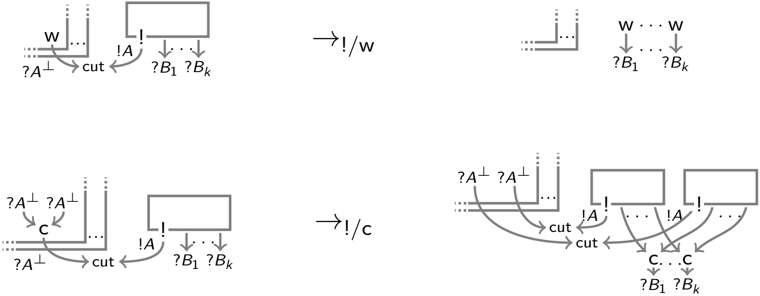
Girard's original presentation of proof nets has a commutative case:



This rule is **the source of all technical complications**.

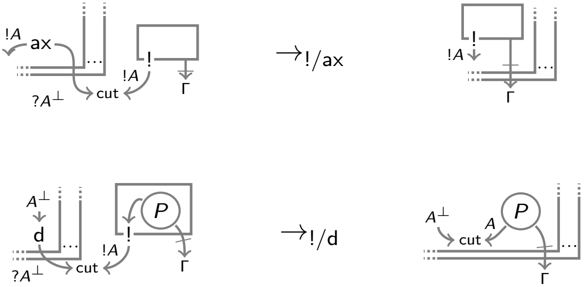
Expliciting implicit cut-elimination

- Implicit boxes provide a new dynamics.
- **Main point**: Box borders are invisible $\Rightarrow$ **No commutative case**.
- Implicit boxes are delicate and **technical**.
- **Idea**: let's rephrase the new dynamics with **explicit boxes**.



The rules act through possibly many box borders.

Box-crossing rules 2

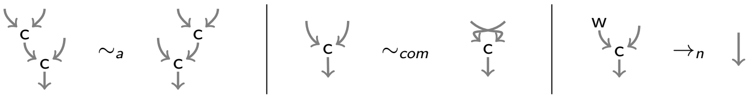


These two cases **absorb the commutative case**.

Box-crossing rules 3

Commutation with box-borders:

Cocommutative comonoid:



Outline

Exponentials and explicit substitutions

- **Statically**:
  In linear logic $A \Rightarrow B$ decomposes as $!A \multimap B$.
- **Dynamically**:
  β splits in a **multiplicative cut** followed by an **exponential cut**.
- **Intuition**: exponentials = **explicit substitutions**.
- **Ordinary substitution** or **implicit substitution**: $t\{x/s\}$.
- **Explicit substitution**: $t[x/s]$.
- **Then**:

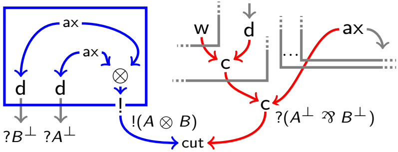
$$(\lambda x.t)s \rightarrow_\beta t\{x/s\}$$

becomes

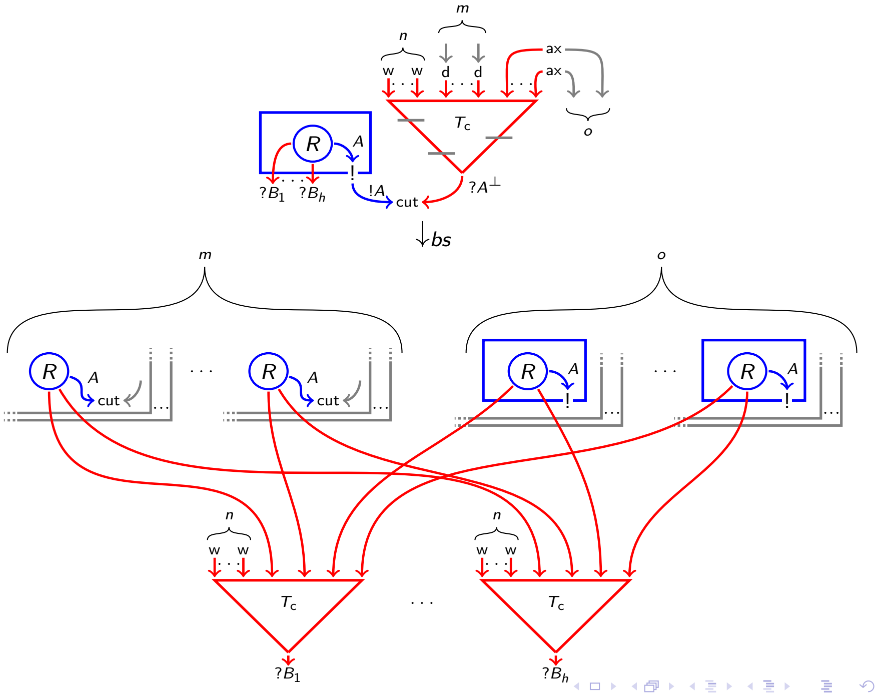
$$(\lambda x.t)s \rightarrow_{\mathtt{m}} t[x/s] \rightarrow_{\mathtt{e}}^{*} t\{x/s\}$$

What is a variable?

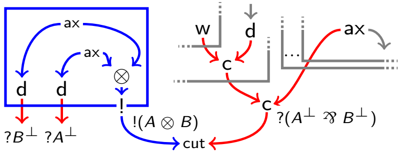
- $[x/s]$ is a **!-box** containing s.
- $t[x/s]$ is a cut between t and the **!-box** around s.
- **What is a variable?** a **maximal tree of ?-rules** (crossing boxes).
- **Example of explicit substitution** $t[x/s]$:



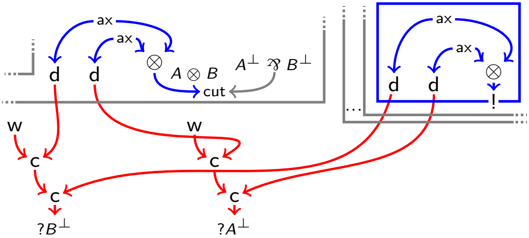
- **Next slide**: definition of substitution in proof nets.


m
n
w w
d d
ax
ax
o
T_c
R
A
!
$?B_1$ $?B_h$
$!A$
cut
$?A^\perp$
$\downarrow bs$
m
o
R A cut
R A cut
R A !
R A !
n
w w
T_c
n
w w
T_c
$?B_1$
$?B_h$

Example of substitution



$\downarrow_{bs}$



Outline

The proof technique

- **Proof technique**:
  Reducibility candidates in bi-orthogonal form (Girard '87).
- **The proof is axiomatic**:
  It works for **every set of rewriting rules** satisfying the axioms.
- For **Girard's rules** the axioms are **hard to prove**.
- For **the rules induced by implicit boxes** the axioms are **easy**.

The axiomatic proof

The proof depends on **3 abstract properties** of the rewriting relation $\to$:

1. **Substitution and promotion commute**:

$$!(P\{x/Q\}) \to^* (!P)\{x/Q\}$$

2. **Full composition**:

$$P[x/Q] \to^+ P\{x/Q\}$$

3. **Kesner's IE property**:

$$\frac{P\{x/Q\} \in SN_\to \qquad Q \in SN_\to}{P[x/Q] \in SN_\to}$$

These properties hold in the **untyped case**.

The IE property

- Key property of λ-calculus:

$$\frac{t\{x/s\}u_1 \ldots u_n \in SN_\beta \qquad s \in SN_\beta}{(\lambda x.t)su_1 \ldots u_n \in SN_\beta}$$

  called **the fundamental lemma of perpetuality** by van Raamsdonk, Severi, Sorensen, and Xi.

- It is more or less explicitly **used in all proofs of SN**, e.g. van Daalen's for simple types, or Girard's for system F.

- Key point in **inductive definitions of the set of SN λ-terms** (van Raamsdonk & Severi, Loader).

- **Kesner**, LMCS '09:
  Preservation of SN for exp. subst. reduces to the **IE property**:

$$\frac{t\{x/s\}u_1 \ldots u_n \in SN_\beta \qquad s \in SN_\beta}{t[x/s]u_1 \ldots u_n \in SN_\beta}$$

Key point of the new proof

- The proof is by **induction on the structure** of the net.
- The difficult case is for **promotion**.
- **Inductive Hypothesis**: $!(P[x/Q]) \in SN_{\rightarrow}$ (and $Q \in SN_{\rightarrow}$).
- **Goal**: $(!P)[x/Q] \in SN_{\rightarrow}$.
- **Key point of the proof**:

$$\begin{array}{rclll} !(P[x/Q]) & \rightarrow^{+} & !(P\{x/Q\}) \in SN & \text{by full composition and i.h.} \\ & \rightarrow^{*} & (!P)\{x/Q\} \in SN & \text{by commutation} \\ & \text{implies} & (!P)[x/Q] \in SN & \text{by the IE property} \end{array}$$

- **Novelty**: no analysis of the reducts of $!(P[x/Q])$.

Confluence

- Main difficulty for the additives: **they are not confluent**.
- **All previous proofs of SN use confluence**.
- That's why T. de Falco and Pagani's proof is **very technical**.
- **Here**: the **first proof** of SN **not requiring confluence**.
- **Consequence**: it smoothly **scales up** to the additives.

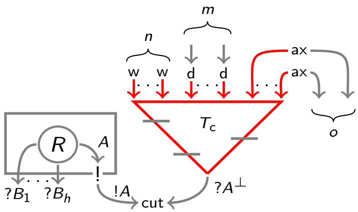
Outline

Proving the IE property 1

The proof of **the IE property**:

1. box-crossing rules: **two lemmas**, a simple induction on a triple.
2. black-box rules: **many lemmas and pages**, *very* technical.

Recall the possible interactions with a graphical variable/?-tree:



	base cases	inductive cases
box-crossing:	ax, der, weak	contraction
black-box	ax, der, weak	contraction, commutative

Proving the IE property 2

- The **IE property**:

$$\frac{P\{x/Q\} \in SN_{\rightarrow} \qquad Q \in SN_{\rightarrow}}{P[x/Q] \in SN_{\rightarrow}}$$

- The proof is by induction on a **triple**:

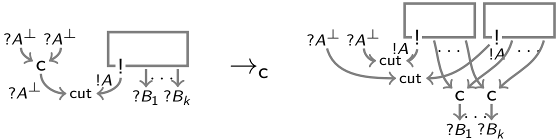
$$(\eta(P\{x/Q\}), |T_x|, \eta(Q))$$

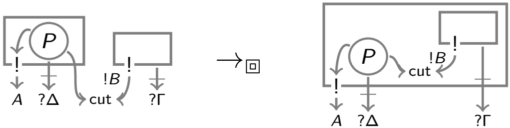
  where $\eta(P)$ is the sum of the lengths of reductions from P.

- Actually, everything has to be generalized to n **substitutions**.

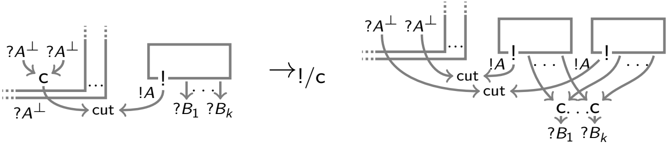
Comparing inductive cases

Black-box rules:





Box-crossing rule:



Intuition: the commutative rule **breaks** the explicit substitution form.

Commutation of promotion and substitution

The property:

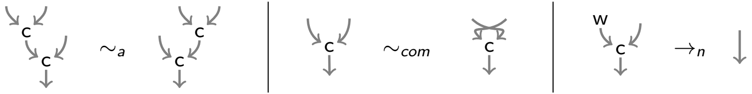
$$!(P\{x/Q\}) \rightarrow^* (!P)\{x/Q\}$$

It follows immediately from **commutation with box borders**:

These rules are **semantically sound** and **needed** to represent λ-terms.

?-Comonoids

- The rules for **?-commutative comonoids**:



are not in the system studied by Tortora de Falco and Pagani.

- Usually, their addition requires **delicate and sophisticated reasoning** (Di Cosmo & Guerrini, Tranquilli & Pagani).
- Here their treatment is almost **transparent**.

Ideas

- **Kesner, LMCS '09**:
  IE technique for SN of explicit substitutions.
- **A.-Guerrini, CSL '09**:
  box-free PN for λ-terms with explicit substitutions.
- **A.-Kesner, CSL '10**:
  1. **new approach** to explicit substitutions (structural λ-calculus λ_{j}).
  2. IE technique **applies extremely easily to λ_{j}**.
- **Here, RTA '13**:
  back to PN, generalizing Kesner's technique and its application.

Conclusions 1

- An **alternative representation of boxes**:
  - Simple;
  - Canonical;
  - More parallel;
  - Provided of a correctness criterion;
  - A local reconstruction of boxes;
  - Partial results on the dynamics.
- **New perspective on polarity**.

Conclusions 2

1. A neat understanding of **substitution** for proof nets (PN).
2. A simple **axiomatic proof of strong normalization** for LL.
3. A **new presentation of PN** s.t. the axioms are easy to verify.
4. A **new understanding of cut-elimination** and exponential boxes.
5. A **fruitful interaction** between **LL** and **explicit substitutions**.

THANKS!