Realizability games for the specification problem

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The *manège enchanté*

Curry-Howard correspondence

Proof theory	**Functional programming**
Proposition	Type
Deduction rule	Typing rule
$A \Rightarrow B$	$A \rightarrow B$
$\dfrac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B}$	$\dfrac{\Gamma \vdash t : A \rightarrow B \qquad \Gamma \vdash u : A}{\Gamma \vdash (t)u : B}$

- Constructive mathematics: intuitionistic logic
- Correct (for the execution) program might be untypable :

```
let stupid n =
        if n=n+1 then 27 else true
```

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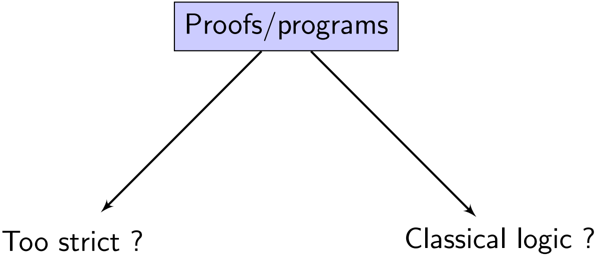
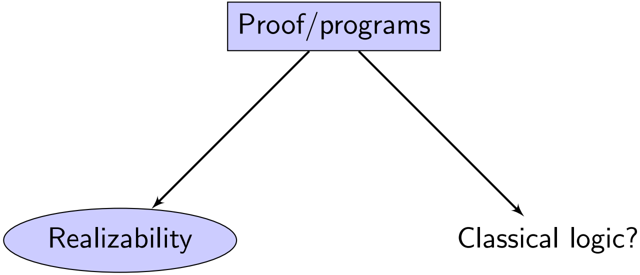


Relaxing: realizability



Relaxing: realizability

Realizers

- $t \Vdash \text{Nat}$ si $t \succ \overline{n}$
- $t \Vdash A \Rightarrow B$ si $u \Vdash A$ implies $(t)u \Vdash B$

- Definition purely computational: no sintax
- Relation $t \Vdash A$ undecidable

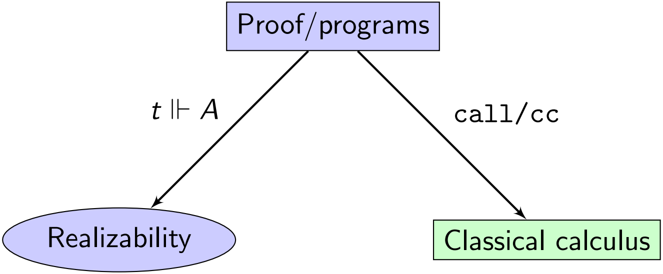
Classical logic

Griffin, 1990

$$\texttt{call/cc} : ((A \Rightarrow B) \Rightarrow A) \Rightarrow A$$

- Intuitionistic logic + Peirce's Law = Classical logic
- Classical Curry-Howard :
  ↬ add a control operator + its typing rule
- Backtrack makes computational analysis harder

Classical realizability



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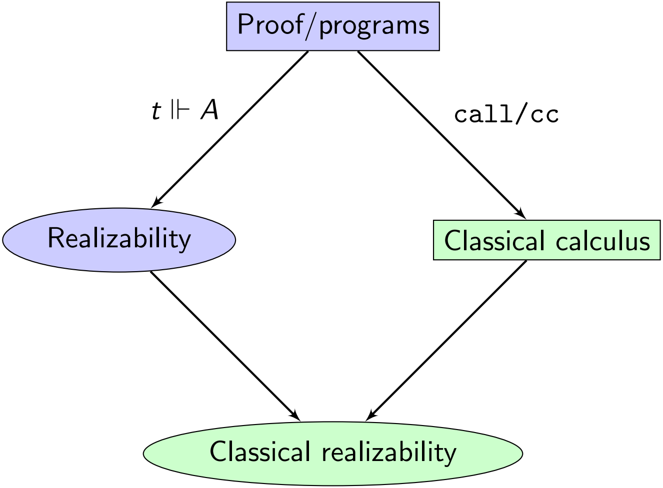

The question of this talk

Specification of A:

Can we give a **characterization** of the realizers of A ?

Yet another introduction to

Krivine classical realizability

λ_c-calculus

Terms, stacks, processes

$\mathcal{B}$: stack constants

$\mathcal{C}$: instructions (including **cc**), countable

Terms	t, u	$::=$	$x \mid \lambda x.t \mid tu \mid \mathbf{k}_\pi \mid \kappa$	$\kappa \in \mathcal{C}$
Stacks	π	$::=$	$\alpha \mid t \cdot \pi$	$(\alpha \in \mathcal{B}, t$ closed$)$
Processes	p, q	$::=$	$t \star \pi$	$(t$ closed$)$

KAM

:

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KAM

$$\begin{array}{rrcl} \textsc{Push}: & (t)u \star \pi & \succ_1 & t \star u \cdot \pi \\ \textsc{Grab}: & \lambda x.t \star u \cdot \pi & \succ_1 & t\{x := u\} \star \pi \\ \vdots & & & \end{array}$$

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Grab	:	$\lambda x.t \star u \cdot \pi$	$\succ_1$	$t\{x := u\} \star \pi$
Save	:	$\mathbf{cc} \star t \cdot \pi$	$\succ_1$	$t \star \mathbf{k}_\pi \cdot \pi$
Restore	:	$\mathbf{k}_\pi \star t \cdot \rho$	$\succ_1$	$t \star \pi$

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KAM $+ \mathcal{C}$ extended

Save	:	$\mathbf{cc} \star t \cdot \pi$	$\succ_1$	$t \star \mathbf{k}_\pi \cdot \pi$
Quote	:	$\texttt{quote} \star \phi \cdot t \cdot \pi$	$\succ_1$	$t \star \overline{n_\phi} \cdot \pi$
Fork	:	$\pitchfork \star t \cdot u \cdot \pi$	$\succ_1$	$t \star \pi$
Fork	:	$\pitchfork \star t \cdot u \cdot \pi$	$\succ_1$	$u \star \pi$
Print	:	$\texttt{print} \star \overline{n} \cdot t \cdot \pi$	$\succ_1$	$t \star \pi$

2^{nd}-order arithmetic

Language

Expressions $\quad e \quad ::= \quad x \mid f(e_1, \ldots, e_k)$

Formulæ $\quad A, B \quad ::= \quad X(e_1, \ldots, e_k) \mid A \Rightarrow B \mid \forall x A \mid \forall X A$

Shorthands :

$$
\begin{aligned}
\bot &\equiv \forall Z.Z \\
\neg A &\equiv A \Rightarrow \bot \\
A \wedge B &\equiv \forall Z((A \Rightarrow B \Rightarrow Z) \Rightarrow Z) \\
A \vee B &\equiv \forall Z((A \Rightarrow Z) \Rightarrow (B \Rightarrow Z) \Rightarrow Z) \\
A \Leftrightarrow B &\equiv (A \Rightarrow B) \wedge (B \Rightarrow A) \\
\exists x A(x) &\equiv \forall Z(\forall x(A(x) \Rightarrow Z) \Rightarrow Z) \\
\exists X A(X) &\equiv \forall Z(\forall X(A(X) \Rightarrow Z) \Rightarrow Z) \\
e_1 = e_2 &\equiv \forall Z(Z(e_1) \Rightarrow Z(e_2))
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Typing rules

$$\frac{}{\Gamma \vdash x : A}\ (x : A) \in \Gamma \qquad \frac{}{\Gamma \vdash t : \top}\ FV(t) \subset \mathrm{dom}(\Gamma)$$

$$\frac{\Gamma \vdash t : A \Rightarrow B \qquad \Gamma \vdash t : A}{\Gamma \vdash tu : B} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \Rightarrow B}$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t : \forall x.A}\ x \notin FV(\Gamma) \qquad \frac{\Gamma \vdash t : \forall x.A}{\Gamma \vdash t : A\{x := e\}}$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t : \forall X.A}\ X \notin FV(\Gamma) \qquad \frac{\Gamma \vdash t : \forall X.A}{\Gamma \vdash t : A\{X(x_1, \dots, x_k) := B\}}$$

$$\frac{}{\Gamma \vdash \mathbf{cc} : ((A \Rightarrow B) \Rightarrow A) \Rightarrow A}$$

Semantics

Intuition

- falsity value $\|A\|$: stacks, opponent to A
- truth value $|A|$: terms, player of A
- pole $\perp\!\!\!\perp$: processes, referee

$$t \star \pi \succ p_0 \succ \cdots \succ p_n \in \perp\!\!\!\perp ?$$

$\rightsquigarrow \perp\!\!\!\perp \subset \Lambda_c \star \Pi$ closed by anti-reduction

Truth value defined by **orthogonality** :

$$|A| = \|A\|^{\perp\!\!\!\perp} = \{t \in \Lambda_c : \forall \pi \in \|A\|, t \star \pi \in \perp\!\!\!\perp\}$$

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Models $(\mathcal{M}, \perp\!\!\!\perp)$

Ground model $\mathcal{M}$

Pole

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$$\forall p, p' \in \Lambda_c \star \Pi : \quad (p \succ p') \wedge (p' \in \perp\!\!\!\perp) \Rightarrow p \in \perp\!\!\!\perp$$

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- $\|\forall x A\| = \bigcup_{n \in \mathbb{N}} \|A\{x := n\}\|$
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Notation

$$t \Vdash A \quad \text{iff} \quad t \in |A| = \|A\|^{\perp\!\!\!\perp}$$
$$t \Vvdash A \quad \text{iff} \quad t \Vdash A \ \text{ for all } \perp\!\!\!\perp$$

Remarks

Case $⫫ = \emptyset$ (degenerated model)

- Truth as in the standard model:

$$|A| = \begin{cases} \Lambda & \text{if } [\![A]\!] = 1 \\ \emptyset & \text{if } [\![A]\!] = 0 \end{cases}$$

- Realizable $\Leftrightarrow$ True in the standard model

Case $⫫ \neq \emptyset$

- $t \star \pi \in ⫫ \Rightarrow$ forall A, $\mathbf{k}_\pi t \Vdash A$
- Restriction to **proof-like**

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Properties

Realizing Peano axioms

If $PA2 \vdash A$, then there is a closed proof-like term t s.t. $t \Vdash A$.

Witness extraction

If $t \Vdash \exists^{\mathbb{N}} x\, A(x)$ and $A(x)$ is atomic or decidable, then we can build a term u s.t. that $\forall \pi \in \Pi$:

$$t \star u \cdot \pi \quad \succ \quad \texttt{stop} \star \overline{n} \cdot \pi \qquad \wedge \qquad A(n) \text{ holds}$$

Adequacy

$$\begin{cases} x_1 : A_1, \ldots, x_k : A_k \vdash t : A \\ \forall i \in [1,k](t_i \Vdash A_i) \end{cases} \quad \Rightarrow \quad t[t_1/x_1, \ldots, t_k/x_k] \Vdash A$$

A Michelin-like metaphor

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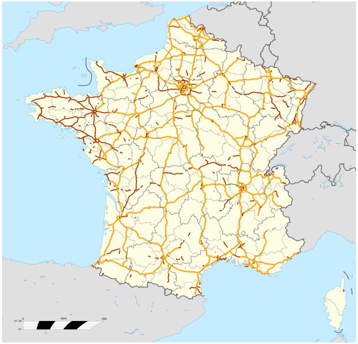

Typing

Realizability

Relativization

$$\mathrm{Nat}(x) \;\equiv\; \forall Z\,(Z(0) \Rightarrow \forall y\,(Z(y) \Rightarrow Z(s(y))) \Rightarrow Z(x))$$

Proposition

There is no $t \in \Lambda_c$ such that $t \Vdash \forall n.Nat(n)$

Better :

$$A, B ::= \;\dots \mid \{e\} \Rightarrow A$$
$$\|\{e\} \Rightarrow A\| \;=\; \{\bar{n}\cdot\pi : [\![e]\!] = n \wedge \pi \in \|A\|\}$$
$$\forall^{\mathrm{N}} x\, A(x) \;\equiv\; \forall x\,(\{x\} \Rightarrow A(x))$$

Let T be a storage operator. The following holds for any formula $A(x)$:

1. $\lambda x.x \Vdash \forall^{nat} x.A(x) \Rightarrow \forall^{\mathrm{N}} x.A(x)$
2. $\lambda x.Tx \Vdash \forall^{\mathrm{N}} x.A(x) \Rightarrow \forall^{nat} x.A(x)$

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There is no $t \in \Lambda_c$ such that $t \Vdash \forall n.Nat(n)$

Fix:

$$\forall^{nat} xA := \forall x(\mathrm{Nat}(x) \Rightarrow A)$$

Obviously, $\lambda x.x \Vdash \forall^{nat} x\mathrm{Nat}(x)$

Better :

$$A, B ::= \;\ldots \mid \{e\} \Rightarrow A$$
$$\|\{e\} \Rightarrow A\| \;=\; \{\bar{n}\cdot\pi : [\![e]\!] = n \wedge \pi \in \|A\|\}$$
$$\forall^{\mathrm{N}} x\, A(x) \;\equiv\; \forall x\,(\{x\} \Rightarrow A(x))$$

Let T be a storage operator. The following holds for any formula $A(x)$:

Relativization

$$\mathrm{Nat}(x) \;\equiv\; \forall Z\,(Z(0) \Rightarrow \forall y\,(Z(y) \Rightarrow Z(s(y))) \Rightarrow Z(x))$$

Proposition

There is no $t \in \Lambda_c$ such that $t \Vdash \forall n.Nat(n)$

Better :

$$\begin{array}{rcl} A, B & ::= & \ldots \mid \{e\} \Rightarrow A \\ \|\{e\} \Rightarrow A\| & = & \{\bar{n} \cdot \pi : [\![e]\!] = n \wedge \pi \in \|A\|\} \\ \forall^{\mathrm{N}} x\, A(x) & \equiv & \forall x\,(\{x\} \Rightarrow A(x)) \end{array}$$

Let T be a storage operator. The following holds for any formula $A(x)$:

1. $\lambda x.x \Vdash \forall^{nat} x.A(x) \Rightarrow \forall^{\mathrm{N}} x.A(x)$
2. $\lambda x.Tx \Vdash \forall^{\mathrm{N}} x.A(x) \Rightarrow \forall^{nat} x.A(x)$

A short digression through models

- Initially designed for PA^2, but we can design model of ZF, and in particular simulate Cohen's forcing.
- Remember there is no $t \Vdash \forall x \mathrm{Nat}(x)$? In fact, there is $\perp\!\!\!\perp$ s.t.:

$$(\mathcal{M}, \perp\!\!\!\perp) \Vdash \exists x \neg \mathrm{Nat}(x)$$

- As a "consequence", we can build a model of ZF in which $\mathbb{R}$ has some "pathological" subsets $\daleth_n$:
  - $\daleth_2$ is not well-ordered
  - $\daleth_n \hookrightarrow \daleth_{n+1}$
  - $\daleth_{n+1} \not\twoheadrightarrow \daleth_n$
  - $\daleth_m \times \daleth_n \equiv \daleth_{mn}$
- some kind of non-commutative forcing : more power ?

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Our problem

Specification of A

Can we give a characterization of $\{t \in \Lambda_c : t \Vdash A\}$?

Absoluteness

Are arithmetical formulæ absolute for realizability models $(\mathcal{M}, \perp\!\!\!\perp)$?

The specification problem

A first example of specification

Two ways of building poles from any set P of processes.

- goal-oriented :

$$\perp\!\!\!\perp \equiv \{p \in \Lambda_c \star \Pi : \exists p' \in P,\ p \succ p'\}$$

- thread-oriented :

Thread of a process p

$$th_p = \{p' \in \Lambda_c \star \Pi : p \succ p'\}$$

$$\perp\!\!\!\perp \equiv (\bigcup_{p \in P} th_p)^c \equiv \bigcap_{p \in P} th_p^c$$

Ex. on board:

$t \Vdash \forall X.(X \Rightarrow X)$ if and only if $\forall k \forall \pi (t \star k \cdot \pi \succ k \star \pi)$

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$t_0 \Vdash \forall X.(X \Rightarrow X) \Rightarrow X \Rightarrow X$ iff ???

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\begin{array}{rcl}
t_0 \star \kappa_s \cdot \kappa_z \cdot \pi & \succ & \kappa_s \star t_1 \cdot \pi \\
t_1 \star \pi & \succ & \kappa_s \star t_2 \cdot \pi \\
 & \vdots & \\
t_i \star \pi & \succ & \kappa_s \star t_{i+1} \cdot \pi \\
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1. Define $p_i := t_i \star \pi$, $\perp\!\!\!\perp_i := \bigcap_{j \in [0,i]} (th(p_j))^c$ and $\|X\| = \{\pi\}$:
   - $\kappa_z \Vdash_i X$ implies $\kappa_s \not\Vdash_i X \Rightarrow X$ and $p_i \succ \kappa_s \star t_{i+1} \cdot \pi$
   - $\kappa_z \not\Vdash_i X$ implies $p_i \succ \kappa_z \star \pi'$

Termination:
If $\forall i \in \mathbb{N}(\kappa_z \not\Vdash_i X)$, define $\perp\!\!\!\perp_\infty := \bigcap_{i \in \mathbb{N}} (th(p_i))^c$, get a contradiction.

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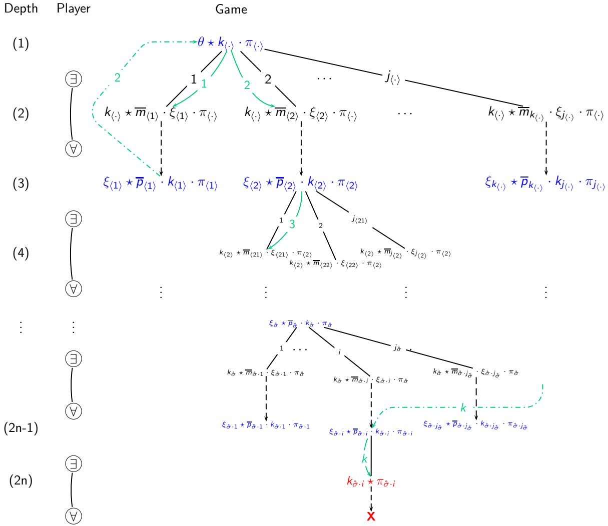
Arithmetical formulæ by hand



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Problem

You want to specify A.

Methodology:

↬ requirement: some intuition...

1. **direct-style**: define the good poles,
2. **indirect-style**: try the thread method,
3. **induction-style**: define a game (brand new!)

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A first notion of game

Coquand's game

Arithmetical formula

$$\Phi_{2h} : \exists x_1 \forall y_1 \ldots \exists x_h \forall y_h f(\vec{x}_h, \vec{y}_h) = 0$$

Rules:

- **Players** : Eloise $\textcircled{\exists}$ and Abelard $\textcircled{\forall}$.
- **Moves** : - at his turn, each player instantiates his variable
  - Eloise allowed to backtrack
- **Final position** : evaluation of $f(\vec{m}_h, \vec{n}_h) = 0$:
  - true : Eloise wins
  - false : game continues
- Abelard wins if the game never ends

Winning strategy

Way of playing that ensures the victory, independently of the opponent moves.

Coquand's game

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Example

Formula

$$\exists x \forall y \exists z (x \cdot y = 2 \cdot z)$$

Player	Action	Position
	Start	$P_0 = (\cdot, \cdot, \cdot)$

Example

Formula

$$\forall y \exists z (1 \cdot y = 2 \cdot z)$$

Player	Action	Position
	Start	$P_0 = (\cdot, \cdot, \cdot)$
$\exists$	$x := 1$	$P_1 = (1, \cdot, \cdot)$

Example

Formula

$$\exists z(1 = 2 \cdot z)$$

Player	Action	Position
	Start	$P_0 = (\cdot,\cdot,\cdot)$
∃ (circled)	$x := 1$	$P_1 = (1,\cdot,\cdot)$
∀ (circled)	$y := 1$	$P_2 = (1,1,\cdot)$

Example

Formula

$$\forall y \exists z (2 \cdot y = 2 \cdot z)$$

Player	Action	Position
	Start	$P_0 = (\cdot, \cdot, \cdot)$
$\exists$	$x := 1$	$P_1 = (1, \cdot, \cdot)$
$\forall$	$y := 1$	$P_2 = (1, 1, \cdot)$
$\exists$	backtrack to P_0 + $x := 2$	$P_3 = (2, \cdot, \cdot)$

Example

Formula

$$\exists z(2 = 2 \cdot z)$$

Player	Action	Position
	Start	$P_0 = (\cdot,\cdot,\cdot)$
∃	$x := 1$	$P_1 = (1,\cdot,\cdot)$
∀	$y := 1$	$P_2 = (1,1,\cdot)$
∃	backtrack to $P_0 + x := 2$	$P_3 = (2,\cdot,\cdot)$
∀	$y := 1$	$P_4 = (2,1,\cdot)$

Example

Formula

$$2 = 2$$

Player	Action	Position
	Start	$P_0 = (\cdot,\cdot,\cdot)$
$\exists$	$x := 1$	$P_1 = (1,\cdot,\cdot)$
$\forall$	$y := 1$	$P_2 = (1,1,\cdot)$
$\exists$	backtrack to $P_0 + x := 2$	$P_3 = (2,\cdot,\cdot)$
$\forall$	$y := 1$	$P_4 = (2,1,\cdot)$
$\exists$	$z := 1$	$P_5 = (2,1,1)$

Example

Formula

$$2 = 2$$

Player	Action	Position
	Start	$P_0 = (\cdot, \cdot, \cdot)$
$\exists$	$x := 1$	$P_1 = (1, \cdot, \cdot)$
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$\forall$	$y := 1$	$P_4 = (2, 1, \cdot)$
$\exists$	$z := 1$	$P_5 = (2, 1, 1)$
	evaluation	$\exists$ wins

Example

Formula

$$2 = 2$$

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	Start	$P_0 = (\cdot, \cdot, \cdot)$
$\exists$	$x := 1$	$P_1 = (1, \cdot, \cdot)$
$\forall$	$y := 1$	$P_2 = (1, 1, \cdot)$
$\exists$	backtrack to P_0 + $x := 2$	$P_3 = (2, \cdot, \cdot)$
$\forall$	$y := 1$	$P_4 = (2, 1, \cdot)$
$\exists$	$z := 1$	$P_5 = (2, 1, 1)$
	evaluation	$\exists$ wins

History

$$H := \bigcup_n P_n$$

$\mathbb{G}^0$: deductive system

Rules:

- If there exists $(\vec{m}_h, \vec{n}_h) \in H$ such that $\mathcal{M} \vDash f(\vec{m}_h, \vec{n}_h) = 0$:

$$\frac{}{H \in \mathbb{W}^0_\Phi}\ \textsc{Win}$$

- For all $i < h$, $(\vec{m}_i, \vec{n}_i) \in H$ and $m \in \mathbb{N}$:

$$\frac{H \cup \{(\vec{m}_i \cdot m, \vec{n}_i \cdot n)\} \in \mathbb{W}^0_\Phi \quad \forall n \in \mathbb{N}}{H \in \mathbb{W}^0_\Phi}\ \textsc{Play}$$

$\mathbb{G}^1$: playing with realizability

Formulæ structure

$$\Phi \equiv \exists^{\mathbb{N}} x_1 \forall^{\mathbb{N}} y_1 \ldots \exists^{\mathbb{N}} x_h \forall y_h (f(\vec{x}_h, \vec{y}_h) = 0)$$
$$\equiv \forall X_1 (\forall^{\mathbb{N}} x_1 (\forall^{\mathbb{N}} y_1 \Phi_1 \Rightarrow X_1) \Rightarrow X_1)$$

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$$\Phi_0 \equiv \forall X_1 (\forall^{\mathbb{N}} x_1 (\forall^{\mathbb{N}} y_1 \Phi_1 \Rightarrow X_1) \Rightarrow X_1)$$

$$\Phi_{i-1} \equiv \forall X_i (\forall^{\mathbb{N}} x_i (\forall^{\mathbb{N}} y_i \Phi_i \Rightarrow X_i) \Rightarrow X_i)$$

$$\Phi_h \equiv \forall W (W(f(\vec{x}_h, \vec{y}_h)) \Rightarrow W(0))$$

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Realizability

$$\|A \Rightarrow B\| = \{u \cdot \pi : u \in |A| \wedge \pi \in \|B\|\}$$
$$\|\forall^{\mathbb{N}} x A(x)\| = \bigcup_{n \in \mathbb{N}} \{\overline{n} \cdot \pi : \pi \in \|A(n)\|\}$$

$\mathbb{G}^1$: playing with realizability

Formulæ structure

$$\Phi_0 \equiv \forall X_1 (\forall^{\mathbb{N}} x_1 (\forall^{\mathbb{N}} y_1 \Phi_1 \Rightarrow X_1) \Rightarrow X_1)$$

Start :

- Eloise proposes t_0 to defend Φ_0
- Abelard proposes $u_0 \cdot \pi_0$ to attack Φ_0

move	p_i ($\exists$-position)	history
0	$t_0 \star u_0 \cdot \pi_0$	$H_0 := \{(\emptyset, \emptyset, u_0, \pi_0)\}$

$\mathbb{G}^1$: playing with realizability

Formulæ structure

$$\Phi_0 \equiv \forall X_1(\forall^{\mathrm{N}} x_1(\forall^{\mathrm{N}} y_1 \Phi_1 \Rightarrow X_1) \Rightarrow X_1)$$

move	p_i ($\exists$-position)	history
0	$t_0 \star u_0 \cdot \pi_0$	$H_0 := \{(\emptyset, \emptyset, u_0, \pi_0)\}$

Eloise reduces p_0 until

- $p_0 \succ u_0 \star \overline{m_1} \cdot t_1 \cdot \pi_0$
  - ↬ she *can* decide to play (m_1, t_1) and ask for Abelard's answer
  - ↬ Abelard *must* give $\overline{n_1} \cdot u' \cdot \pi'$.

$\mathbb{G}^1$: playing with realizability

Formulæ structure

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move	p_i ($\exists$-position)	history
0	$t_0 \star u_0 \cdot \pi_0$	$H_0 := \{(\emptyset, \emptyset, u_0, \pi_0)\}$
1	$t_1 \star \overline{n}_1 \cdot u_1 \cdot \pi_1$	$H_1 := \{(m_1, n_1, u_1, \pi_1)\} \cup H_0$

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- $p_0 \succ u_0 \star \overline{m_1} \cdot t_1 \cdot \pi_0$
  - ↬ she *can* decide to play (m_1, t_1) and ask for Abelard's answer
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$\mathbb{G}^1$: playing with realizability

Formulæ structure

$$\Phi_{i-1} \equiv \forall X_i(\forall^{\mathbb{N}} x_i(\forall^{\mathbb{N}} y_i \Phi_i \Rightarrow X_i) \Rightarrow X_i)$$

move	p_i ($\exists$-position)	history
1	$t_1 \star \overline{n}_1 \cdot u_1 \cdot \pi_1$	$H_1 := \{(m_1, n_1, u_1, \pi_1)\} \cup H_0$
$\vdots$	$\vdots$	$\vdots$
i	$t_i \star \overline{n}_i \cdot u_i \cdot \pi_i$	$H_i := \{(m_i, n_i, u_i, \pi_i)\} \cup H_{i-1}$

Eloise reduces p_i until

- $p_i \succ u \star \overline{m} \cdot t \cdot \pi$ with $(\vec{m}_j, \vec{n}_j, u, \pi) \in H_j$ where $j < h$.
  - ↬ she *can* decide to play (m_{i+1}, t_{i+1})
  - ↬ Abelard *must* give $\overline{n_i} \cdot u' \cdot \pi'$.

$\mathbb{G}^1$: playing with realizability

Formulæ structure

$$\Phi_h \equiv \forall W(W(f(\vec{x}_h, \vec{y}_h)) \Rightarrow W(0))$$

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1	$t_1 \star \overline{n}_1 \cdot u_1 \cdot \pi_1$	$H_1 := \{(m_1, n_1, u_1, \pi_1)\} \cup H_0$
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  - ↬ she *can* decide to play (m_{i+1}, t_{i+1})
  - ↬ Abelard *must* give $\overline{n_i} \cdot u' \cdot \pi'$.
- $p_i \succ u \star \pi$ with $(\vec{m}_h, \vec{n}_h, u, \pi) \in H_j$
  - ↬ she wins iff $\mathcal{M} \models f(\vec{m}_h, \vec{n}_h) = 0$.

$\mathbb{G}^1$: formal definition

- if $\exists(\vec{m}_h, \vec{n}_h, u, \pi) \in H$ s.t. $p \succ u \star \pi$ and $\mathcal{M} \vDash f(\vec{m}_h, \vec{n}_h) = 0$:

$$\frac{}{\langle p, H\rangle \in \mathbb{W}^1_\Phi}\ \textsc{Win}$$

- for every $(\vec{m}_i, \vec{n}_i, u, \pi) \in H$, $m \in \mathbb{N}$ s.t. $p \succ u \star \overline{m} \cdot t \cdot \pi$:

$$\frac{\langle t \star \overline{n} \cdot u' \cdot \pi', H \cup \{(\vec{m}_i \cdot m, \vec{n}_i \cdot n, u', \pi')\}\rangle \in \mathbb{W}^1_\Phi \quad \forall(n', u', \pi')}{\langle p, H\rangle \in \mathbb{W}^1_\Phi}\ \textsc{Play}$$

Winning strategy

$t \in \Lambda_c$ s.t. for any handle $(u, \pi) \in \Lambda \times \Pi$:

$$\langle t \star u \cdot \pi, \{(\emptyset, \emptyset, u, \pi)\}\rangle \in \mathbb{W}^1_\Phi$$

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Specification result

Adequacy

If t is a winning strategy for $\mathbb{G}^1_\Phi$, then $t \Vdash \Phi$

Proof (sketch):
- play a match with stacks from falsity value,
- conclude by anti-reduction.

Specification result

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Completeness of $\mathbb{G}^1$

If the calculus is deterministic and substitutive, then if $t \Vdash \Phi$ then t is a winning strategy for the game $\mathbb{G}^1_\Phi$

Proof (sketch): by contradiction
- substitute Abelard's winning answers along the thread scheme,
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The general case

Loosing the substition

quote

$$\texttt{quote} \star \varphi \cdot t \cdot \pi \succ t \star \overline{n_\varphi} \cdot \pi$$

- the calculus is no longer substitutive
- there are some wild realizers which are not winning strategies!

Consider $\Phi_\leq \equiv \exists^{\mathbb{N}} x \forall^{\mathbb{N}} y (x \leq y)$ and $t_\leq$ s.t. :

$$t_\leq \star u \cdot \pi \succ T_0 \star \pi \succ u \star \bar{0} \cdot T_1 \cdot \pi$$

and :

$$T_1 \star \bar{n} \cdot u' \cdot \pi' \succ \begin{cases} \mathbf{I} \star \pi' & \text{if } u' \equiv T_0 \text{ and } \pi \equiv \pi' \\ u' \star \pi' & \text{otherwise} \end{cases}$$

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$\rightsquigarrow$ Idea : *I've already been there...*

$\mathbb{G}^2$: non-substitutive case

⇝ Idea : *I've already been there...*

- if $\exists(\vec{m}_h, \vec{n}_h, u, \pi) \in H$ s.t. $p \succ u \star \pi$ and $\mathcal{M} \vDash f(\vec{m}_h, \vec{n}_h) = 0$:

$$\frac{}{\langle p, H\rangle \in \mathbb{W}^1_\Phi}\ \textsc{Win}$$

- for every $(\vec{m}_i, \vec{n}_i, u, \pi) \in H$, $m \in \mathbb{N}$ s.t. $p \succ u \star \overline{m} \cdot t \cdot \pi$:

$$\frac{\langle t \star \overline{n} \cdot u' \cdot \pi', H \cup \{(\vec{m}_i \cdot m, \vec{n}_i \cdot n, u', \pi')\}\rangle \in \mathbb{W}^1_\Phi \quad \forall(n', u', \pi')}{\langle p, H\rangle \in \mathbb{W}^1_\Phi}\ \textsc{Play}$$

$\mathbb{G}^2$: non-substitutive case

⇝ Idea : *I've already been there...*

- if $\exists(\vec{m}_h, \vec{n}_h, u, \pi) \in H$, $\exists p \in \mathbf{P}$ s.t. $p \succ u \star \pi$ and $\mathcal{M} \vDash f(\vec{m}_h, \vec{n}_h) = 0$:

$$\frac{}{\langle \mathbf{P}, H\rangle \in \mathbb{W}^2_\Phi}\ \textsc{Win}$$

- for every $(\vec{m}_i, \vec{n}_i, u, \pi) \in H$, $m \in \mathbb{N}$ s.t. $\exists p \in \mathbf{P}$, $p \succ u \star \overline{m} \cdot t \cdot \pi$:

$$\frac{\langle \{t \star \overline{n} \cdot u' \cdot \pi'\} \cup \mathbf{P}\rangle, H \cup \{(\vec{m}_i \cdot m, \vec{n}_i \cdot n, u', \pi')\} \in \mathbb{W}^2_\Phi \quad \forall (n', u', \pi')}{\langle \mathbf{P}, H\rangle \in \mathbb{W}^2_\Phi}\ \mathrm{P}$$

Specification result

Adequacy

If t is a winning strategy for $\mathbb{G}^2_\Phi$, then $t \Vdash \Phi$

Proof (sketch):
- play a match with stacks from falsity value,
- conclude by anti-reduction.

Specification result

Adequacy

If t is a winning strategy for $\mathbb{G}^2_\Phi$, then $t \Vdash \Phi$

Proof (sketch):
- play a match with stacks from falsity value,
- conclude by anti-reduction.

Completeness of $\mathbb{G}^2$

If $t \Vdash \Phi$ then t is a winning strategy for the game $\mathbb{G}^2_\Phi$

Proof (sketch): by contradiction,
- build an increasing sequence $\langle P_i, H_i \rangle$ using $\textcircled{\forall}$ winning answers,
- define $\perp\!\!\!\perp := (\bigcup_{p \in P_\infty} \mathbf{th}(p))^c$,
- reach a contradiction.

Consequences

Proposition: Uniform realizer

There exists a term T such that if:
- $\mathcal{M} \vDash \exists x_1 \forall y_1 \ldots f(\vec{x}, \vec{y}) = 0$
- θ_f computes f

then

$$T\theta_f \Vdash \exists x_1 \forall y_1 \ldots f(\vec{x}, \vec{y}) = 0$$

Proposition

There is a winning strategy iff $\mathcal{M} \vDash \exists x_1 \forall y_1 \ldots f(\vec{x}, \vec{y}) = 0$.

Theorem: Absoluteness

If Φ is an arithmetical formula, then

$$\exists t \in \Lambda_c, t \Vdash \Phi \qquad \text{iff} \qquad \mathcal{M} \vDash \Phi$$

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Comments & conclusion

About equality

$$\Phi_{2h} : \exists x_1 \forall y_1 \ldots \exists x_h \forall y_h f(\vec{x}_h, \vec{y}_h) \neq 0$$

	$\|f(x) = 0\|$	$\|f(x) \neq 0\|$
$\mathcal{M} \vDash f(x) = 0$	$\|\forall X.X \to X\|$	Π
$\mathcal{M} \vDash f(x) \neq 0$	$\Lambda_c \times \Pi$	$\emptyset$

Uniform realizer

$\forall n \in \mathbb{N}$, there exists $t_n \in \Lambda_c$ s.t. $\forall f : \mathbb{N}^{2n} \to \mathbb{N}$,

$$\mathcal{M} \vDash \exists x_1 \forall y_1 \ldots f(\vec{x}, \vec{y}) \neq 0 \quad \Rightarrow \quad t_n \Vdash \exists^{\mathbb{N}} x_1 \forall^{\mathbb{N}} y_1 \ldots f(\vec{x}, \vec{y}) \neq 0.$$

↪ *t does not necessarily play according to the formula...*

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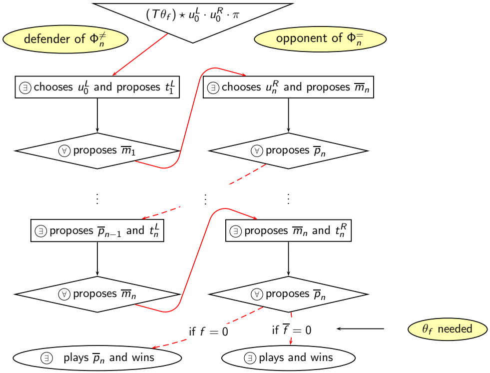
↬ *t does not necessarily play according to the formula...*

Combining strategies

Forall n, there exists a term T_n s.t. if θ_f computes f, then
$$T_n\theta_f \Vdash \Phi_n^{\neq} \Rightarrow \Phi_n^{=}$$



About absoluteness

- it was already known
- it extends to realizability algebras
- we now know even more :

Shoenfield barrier

Every Σ^1_2/Π^1_2-relation is absolute for all *inner models* $\mathcal{M}$ of ZF.

Krivine'14

There exists an ultrafilter on $\beth 2$

Corollary

For any realizability algebra $\mathcal{A}$, $\mathcal{M}^{\mathcal{A}}$ contains a proper class $\mathcal{M}'$ which is an *inner model* of ZF.

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Conclusion

What we did :

- We defined two games for substitutive and non-substitutive cases
- We proved equivalence between universal realizers and winning strategies
- It solved both specification and absoluteness problems

Further work :

- classes of formulæ compatible with games ?
- transformation $\mathbb{G}^1 \rightsquigarrow \mathbb{G}^2$ generic ?
- combination of strategies ?

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Thank you for your attention.