$$(\forall M \in \Lambda_+)\ M \in \mathsf{SN} \iff \mathcal{T}(M) \in \mathfrak{F}_{\mathsf{SN}}$$

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based on joint work with Christine Tasson° and Michele Pagani°
(mainly our FoSSaCS 2016 paper)

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Outline

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We characterize the strong normalizability (SN)
of non-deterministic λ-terms (Λ_+)
as a finiteness structure ($\mathfrak{F}_{\mathsf{SN}}$)
via Taylor expansion ($\mathcal{T}$).

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The end

Thanks for your attention.

Denotational semantics

A very old idea

Terms of type $A \to B$ are functions from A to B.

Denotational semantics

An old idea

types	A	$\rightsquigarrow$	objects	$[\![A]\!]$
terms	$x : A \vdash M : B$	$\rightsquigarrow$	morphisms	$[\![M]\!] : [\![A]\!] \to [\![B]\!]$
substitution	$x : A \vdash N\,[M/y] : C$	$\rightsquigarrow$	composition	$[\![A]\!] \xrightarrow{[\![M]\!]} [\![B]\!] \xrightarrow{[\![N]\!]} [\![C]\!]$
reduction	$M \to_\beta N$	$\rightsquigarrow$	equality	$[\![M]\!] = [\![N]\!]$

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Hopefully, we model something interesting (continuity, stability, etc.).

Quantitative semantics

A prime aged idea (Girard, '80s)

- types $\rightsquigarrow$ particular topological vector spaces:
  $[\![A]\!] \subseteq \mathbf{k}^{|A|}$ + possibly some additional structure
- terms $\rightsquigarrow$ analytic functions defined by power series:

$$
\begin{aligned}
|A \to B| &\subseteq |A|^! \times |B| \\
((M)\,N)_\beta &= \sum_{(\overline{\alpha},\beta)} M_{(\overline{\alpha},\beta)} N^{\overline{\alpha}}
\end{aligned}
$$

  where
  - $|A|^! = \mathfrak{M}_f\left(|A|\right)$
  - $\overline{\alpha} = [\alpha_1, \ldots, \alpha_n] \in |A|^!$, $\beta \in |B|$
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How to ensure the convergence of the series?

Originally, $\mathbf{k} = \mathsf{Sets}$.

Finiteness structures

Definition (Ehrhard, early 2000's)

- If $a, a' \subseteq A$, write $a \perp a'$ iff $a \cap a'$ is finite.
- If $\mathfrak{S} \subseteq \mathfrak{P}(A)$, let $\mathfrak{S}^\perp := \{a' \subseteq A;\ \forall a \in \mathfrak{S},\ a \perp a'\}$.
- A finiteness structure is any $\mathfrak{F} = \mathfrak{S}^\perp$.

Then you can build a denotational model of linear logic where

$$[\![A]\!] = \left\{a \in \mathbf{k}^{|A|};\ |a| \in \mathfrak{Fin}(A)\right\}$$

with $\mathfrak{Fin}(A)$ a finiteness structure on $|A|$ so that for all $a \in \mathfrak{Fin}(A)$, $\beta \in |B|$ and all $f \in \mathfrak{Fin}(A \to B)$,

$$\{\overline{\alpha};\ (\overline{\alpha}, \beta) \in f\} \perp a^!.$$

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Short version

The sum in the previous slide is always finite.

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The sum in the previous slide is always finite.

Moral

Finiteness structures enforce finite interaction/reduction/cut elimination.

λ-terms as analytic functions

So we can *differentiate* (typed) λ-terms, and compute their Taylor expansion!
And one can mimick that in the syntax:

- differential λ-calculus (Ehrhard-Regnier 2003)
- a finitary fragment: resource λ-calculus (Ehrhard-Regnier 2004)
  this is the target of Taylor expansion

Resource λ-calculus

Resource terms

$$\begin{array}{llll} \Delta & \ni & s, t, \ldots & ::= & x \mid \lambda x.s \mid \langle s \rangle\ \bar{t} \\ \Delta^! & \ni & \bar{s}, \bar{t}, \ldots & ::= & [s_1, \ldots, s_n] \end{array}$$

Meaning: $\langle s \rangle\ [s_1, \ldots, s_n] = (Ds)_0 \cdot (s_1, \ldots, s_n)$

Resource reduction

$$\langle \lambda x.s \rangle\ \bar{t} \to_\rho \partial_x s \cdot \bar{t} \quad \text{(anywhere)}$$

$$\partial_x s \cdot \bar{t} = \begin{cases} \sum_{f \in \mathfrak{S}_n} s \left[t_{f(1)}, \ldots, t_{f(n)} / x_1, \ldots, x_n \right] & \text{if } \deg_x(s) = \#\bar{t} = n \\ 0 & \text{otherwise} \end{cases}$$

sums $S, T, \ldots := \sum_{i=1}^n t_i$ with $\lambda x.0 = 0$, $\langle s \rangle\ [t_1 + t_2, u] = \sum_i \langle s \rangle\ [t_i, u]$, …

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- Resource reduction preserves free variables, is size-decreasing, strongly confluent and normalizing.

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- Resource reduction preserves free variables, is size-decreasing, strongly confluent and normalizing.
- Normal forms $\sim$ elements of some relational model.

Taylor expansion of λ-terms

Semantically, $(M)\,N = \sum_{n\in\mathbf{N}} \frac{1}{n!} \langle M \rangle\; N^n$ where $N^n = [N, \ldots, N]$.

Taylor expansion: $\vec{\mathcal{T}}\,(M) \in \mathbf{Q}^{+\Delta}$

$$\vec{\mathcal{T}}\,((M)\,N) = \sum_{n\in\mathbf{N}} \frac{1}{n!} \left\langle \vec{\mathcal{T}}\,(M) \right\rangle \vec{\mathcal{T}}\,(N)^n$$

$$\vec{\mathcal{T}}\,(x) = x \qquad \vec{\mathcal{T}}\,(\lambda x.M) = \lambda x.\vec{\mathcal{T}}\,(M)$$

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Theorem (Ehrhard-Regnier, TCS 2008)

If $M \in \Lambda$ has a normal form, then $\vec{\mathcal{T}}\,(M)$ normalizes .

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Theorem (Ehrhard-Regnier, TCS 2008 + CiE 2006)

If $M \in \Lambda$ has a normal form, then $\vec{\mathcal{T}}\,(M)$ normalizes to $\vec{\mathcal{T}}\,(\mathsf{NF}\,(M))$.

Theorem (Ehrhard-Regnier, CiE 2006)

In general $\vec{\mathcal{T}}\,(M)$ normalizes to $\vec{\mathcal{T}}\,(\mathsf{BT}\,(M))$.

Moral

In the uniform setting $\mathsf{BT}\,(M) \simeq \mathsf{NF}(\vec{\mathcal{T}}\,(M))$.

Normalizing Taylor expansions

But how can $\vec{\mathcal{T}}(M)$ even normalize?

Take $\vec{a} \in \mathbf{k}^{\Delta}$: we want to set

$$\mathsf{NF}(\vec{a}) = \sum_{t \in \Delta} a_t.\mathsf{NF}(t)$$

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Theorem (Ehrhard-Regnier 2004)

Write $\mathcal{T}(M) = \left|\vec{\mathcal{T}}(M)\right|$. *Then for all* $t \in \Delta$, *there is at most one* $s \in \mathcal{T}(M)$ *such that* $\mathsf{NF}(s)_t \neq 0$.

Proof.

λ-terms are uniform (= essentially deterministic). □

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This fails in general

$\mathsf{NF}\left(\sum_{n \in \mathbf{N}} \langle \lambda x.x \rangle^n [y]\right)$? $\qquad$ $\langle \lambda x.x \rangle^n [y] = \langle \lambda x.x \rangle [\langle \lambda x.x \rangle [\cdots [y] \cdots]]$

What about non-deterministic λ-calculi?

A minimalistic non-uniform calculus

$$\Lambda_+ \ni M, N, \ldots ::= x \mid \lambda x.M \mid (M)\,N \mid M + N$$

$$(\lambda x.M)\,N \to_\beta M\,[N/x] \quad \text{(anywhere)}$$

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Example

Let $\delta_M = \lambda x.\,(M + (x)\,x)$ and $\infty_M = (\delta_M)\,\delta_M$:

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Let $\delta_M = \lambda x.\,(M + (x)\,x)$ and $\infty_M = (\delta_M)\,\delta_M$: $\infty_M \to_\beta^* M + \infty_M$!

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Taylor expansion in a non uniform setting

$$\vec{\mathcal{T}}\,(M + N) = \vec{\mathcal{T}}\,(M) + \vec{\mathcal{T}}\,(N)$$

We would like to set:

$$\mathsf{NF}\left(\vec{\mathcal{T}}\,(M)\right) = \sum_{s \in \Delta} \vec{\mathcal{T}}\,(M)_s\,\mathsf{NF}\,(s)$$

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Then $\mathsf{NF}\left(\vec{\mathcal{T}}\,(\infty_M)\right) = ?$

Finiteness structures to the rescue

When is $\mathsf{NF}(\vec{\mathcal{T}}(M))$ defined?

- Write $s \geq t$ if $s \to_\rho^* t + \cdots$.
- Let $\uparrow t = \{s \in \Delta;\ s \geq t\}$.
- We want: for all normal $t \in \Delta$, $\mathcal{T}(M) \perp \uparrow t$.

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Let system F_+ be system F plus $\dfrac{\Gamma \vdash M : A \qquad \Gamma \vdash N : A}{\Gamma \vdash M + N : A}$.

Theorem (Ehrhard, LICS 2010)

If $M \in \Lambda_+$ is typable in system F_+, then $\mathcal{T}(M) \in \{\uparrow t\ ;\ t \in \Delta\}^\perp$.

Proof.

Manage sets of resource terms as if they were λ-terms, and follow the usual reducibility technique, associating a finiteness structure $\mathfrak{Fin}(A) \subseteq \{\uparrow t\ ;\ t \in \Delta\}^\perp$ with each type A.

□

A remark

In the previous theorem, "tests" are not restricted to normal terms.
This rules out looping terms, e.g., $\Omega = (\Delta)\,\Delta$ with $\Delta = \lambda x.\,(x)\,x$:

- consider $\delta_n = \lambda x.\,\langle x\rangle\ [x^n]$;
- then for all $n \in \mathbf{N}$, $\langle \delta_n\rangle\ [\delta_0, \delta_0, \delta_1 \ldots, \delta_{n-1}] \geq \langle \delta_0\rangle\ [] \to_\rho 0$.

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- Typability in F can be relaxed to strong normalizability.
- Then the implication

$$M \in \mathsf{SN} \Rightarrow \mathcal{T}(M) \in \{\uparrow t \; ; \; t \in \Delta\}^{\perp}$$

  can be reversed…
- provided the finiteness $\{\uparrow t \; ; \; t \in \Delta\}^{\perp}$ is refined to a tighter one.

$M \in \mathsf{SN} \Rightarrow \mathcal{T}(M) \in \{\uparrow t\ ;\ t \in \Delta\}^{\perp}$

In the ordinary λ-calculus:

- SN = typability in system D (simple types + $\cap$)
- "any" proof by reducibility for simple types is valid for D

So we:

- introduce a system D_+ of intersection types for non uniform terms
- prove that $M \in \mathsf{SN}$ implies $\Gamma \vdash M : A$ in D_+
- adapt Ehrhard's proof to D_+

System D_+

System D uses the rules:

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash M : B}{\Gamma \vdash M : A \cap B} \qquad \frac{\Gamma \vdash M : A \cap B}{\Gamma \vdash M : A} \qquad \frac{\Gamma \vdash M : A \cap B}{\Gamma \vdash M : B}$$

This is not sufficient here, due to constraints for typing sums:

- observe that $(x + y)\, z = (x)\, z + (y)\, z$
- let $\Gamma = x : A \to B \cap B', y : A \to B \cap B'', z : A$,
- then $\Gamma \vdash (x + y)\, z : B$
- but $x + y$ is not typable in Γ.

We need (a limited amount of) subtyping:

- $A \cap B \preceq A$ and $A \cap B \preceq B$;
- $(A \to B) \cap (A \to C) \preceq A \to (B \cap C)$;
- $A \to B \preceq A' \to B'$ as soon as $A' \preceq A$ and $B \preceq B'$.

$$\frac{\Gamma \vdash M : A \qquad A \preceq B}{\Gamma \vdash M : B}$$

Then the proofs go *almost* as usual.

$\mathcal{T}(M) \in \{\uparrow t\,;\ t \in \Delta\}^{\perp} \Rightarrow M \in \mathsf{SN}$

$\mathcal{T}(M) \in \{\uparrow t \,;\, t \in \Delta\}^{\perp} \not\Rightarrow M \in \mathsf{SN}$

Fails!

Let $\Delta_3 := \lambda x.\,(x)\,x\,x$ and $\Omega_3 := (\Delta_3)\,\Delta_3$, then $\mathcal{T}(\Omega_3) \perp \uparrow s$ for all s.

$\mathcal{T}(M) \in \{\uparrow t \,;\, t \in \Delta\}^{\perp} \not\Rightarrow M \in \mathsf{SN}$

Fails!

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Why?

We ruled out loops, but the divergence of Ω_3 is of another nature.
A diverging λ-term either loops or reduces to terms of arbitrary height.

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Fix: add more tests

- Consider a structure $\mathfrak{S} \subseteq \mathfrak{P}(\Delta)$ and let $\mathfrak{F}_{\mathfrak{S}} = \{\uparrow a \,;\, a \in \mathfrak{S}\}^{\perp}$ with $\uparrow a = \bigcup_{s \in a} \uparrow s$.
- Of course, not all $\mathfrak{S}$ are acceptable, otherwise we reject too many terms (consider $\mathfrak{S} = \mathfrak{P}(\Delta)$).
- We need to rule out unbounded height: it suffices to test against linear terms.

$\mathcal{T}(M) \in \mathfrak{F}_{\mathfrak{S}} \Rightarrow M \in \mathsf{SN}$

... as soon as $\mathfrak{S}$ contains all sets of linear terms.

$\mathcal{T}(M) \in \mathfrak{F}_{\mathfrak{S}} \Rightarrow M \in \mathsf{SN}$

... as soon as $\mathfrak{S}$ contains all sets of linear terms.

We prove the contraposition: given an infinite reduction sequence from M, we find $a \in \mathfrak{S}$ such that $\mathcal{T}(M) \not\leq \uparrow a$.

Lemma

If $M \to_\beta^* N$ then for all $t \in \mathcal{T}(N)$ there is $s \in \mathcal{T}(M)$ such that $s \geq t$.

Proof that $\mathcal{T}(M) \in \mathfrak{F}_{\mathfrak{S}} \Rightarrow M \in \mathsf{SN}$.

- if M reduces to terms of unbounded height:
  - take M_i any term of height $\geq i$ with $M \to_\beta^* M_i$;
  - take $a = \{s_i;\ i \in \mathbf{N}\}$ with $s_i \in \mathcal{T}(M_i)$ a linear resource term
- otherwise M (in fact $\mathcal{T}(M)$) loops and we can follow a looping reduction path backwards (with some care)

□

Glueing everything together

We can adapt the reducibility proof provided $\mathfrak{S}$ satisfies:

- for all $n \in \mathbf{N}$, for all $a \in \mathfrak{S}$, $\{s \in a;\ \mathbf{h}(s) \leq n\}$ is finite.
- some additional, purely technical conditions.

Example

$\mathfrak{B} = \{a \subseteq \Delta;\ \#(a) \text{ is bounded}\}$ where $\#(a) = \{\#(s);\ s \in a\}$ and $\#(s)$ is the maximum size of a bag of arguments in s.

Theorem (PTV, FoSSaCS 2016)

The following three properties are equivalent:

- $M \in \mathsf{SN}$;
- M *is typable in system* D_+;
- $\mathcal{T}(M) \in \mathfrak{F}_{\mathfrak{B}}$.

Conclusion

We are happy.

We have established a nice and novel characterization of SN.

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Are we?

This is intellectually satisfying but the really useful bit is that:

the Taylor expansion of a strongly normalizable term is normalizable

which is a bit frustrating (why strongly?).

Dealing with normalization

- Our construction is parametrized by:
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A similar technique applies for head normalization.

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One last word

Paves the way for a unified notion of Böhm trees in various non uniform settings (quantitative non-determinism, probabilistic stuff, *etc.*).

The end

Thanks for your attention.

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Questions?

About that other story. . .

DRAFT IN PROGRESS

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It is easy to follow β-reduction backwards:

Lemma

If $M \to_\beta N$ and $t \in \vec{\mathcal{T}}(N)$ then there exists $s \in \vec{\mathcal{T}}(M)$ s.t. $s \geq t$.

Moreover it has a nice fineteness property (which we actually used above):

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Write $\vec{a} \Rightarrow_\rho \vec{b}$ if $\vec{a} = \sum_i a_i s_i$, $\vec{b} = \sum_i b_i S'_i$, with $s_i \Rightarrow_\rho S'i$, for all i.

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but this is not well behaved.

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Exercise

. . .

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