Using the Size-Change Principle for checking totality of recursive definitions

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The “Size-Change Principle”

- Relevant instance of the SCP:

```
{ Head = x1::x2; Tail = { Head = x3 ; Tail = x4 } }
     =>
{  Head = Ω::<∞,-2>x4 ; Tail = <∞>x4  }, .<<-2>>
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- Non relevant instance of the SCP:

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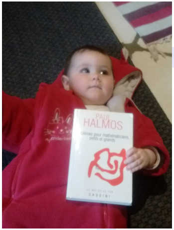

Plan

1. "size-change principle" and inductive types
2. "size-change principle" et productivity
3. "size-change principle" and totality

Goal

- typed functional language
- algebraic datatypes
  - sums (constructors)
  - products (structures)
  - initial algebras (inductive types)
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Termination checker: adaptation of the "size-change principle"
(Lee, Jones et Ben-Amram 2001, P.H. 2014)

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Both functions terminate (on appropriate types)

- `add (m+1) (n+1)` ⇒ `add n (m+1)` ⇒ `add m n`: arguments decrease
- `sum _::(_::l)` ⇒ `sum ?::l`: tail of the argument decreases

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however

- `add m (n+1)` ⇒ `add n m`: no decrease with single call
- `sum n::m::l` ⇒ `sum ((add m n)::l)`: no decrease in whole argument

SCP: idea

Abstract interpretation of recursive call, keeping only

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We get in this way a call graph.

(vertices: mutually defined functions)

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A bunch of mutually defined functions terminate if:

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Composition of path: unification + truncation with 2 parameters

- a depth $\geqslant 0$ for terms (here 3)
- a bound > 0 on coefficients (here 3)

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We get structural recursion in subterms, lexicographic combinations, argument permutations, locale size increase, ...

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- structures are lazy
- the third recursive call isn't guarded (Coquand 1993)
- but the definition is productive

SCP and productivity

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A recursive definition is productive if for all infinite path:

- an 'inductive" branch in an argument is infinite (cf. previous slides),
- the "coinductive" branch of the result is infinite.

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The test is very similar, the coinductive branch of the result is seen as an additional argument.

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data tree where    --  (empty) inductive type
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  | bad_t = Node bad_s
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val bad_t : tree
  | bad_t = Node bad_s
```

- the definition is well-typed (Hindley-Milner)
- the definition is productive
- evaluation of `bad_t` (and all its subterms) terminates
- `bad_t` is not an element of the (empty) type `tree`

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(cf `charity` by R. Cockett)

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totality test: generalizes termination and productivity test
(SCP + "guard conditions" inspired by L. Santocanale's circular proofs)

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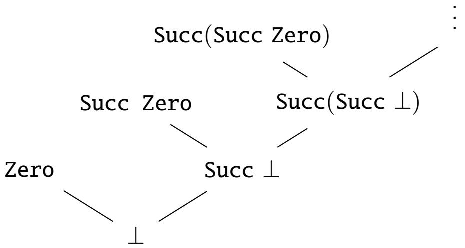

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Definition

A (maximal) element of such a domain is <u>total</u> if it belongs to the corresponding set theoretic interpretation.

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Definition

A (maximal) element of such a domain is total if it belongs to the corresponding set theoretic interpretation.

Goal: find a decidable totality criterion.

Parity games and totality

Coinductive "Rose trees":

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Parity games and totality

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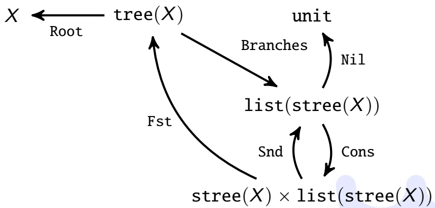


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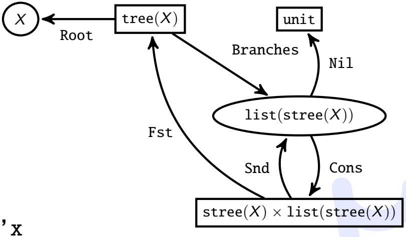


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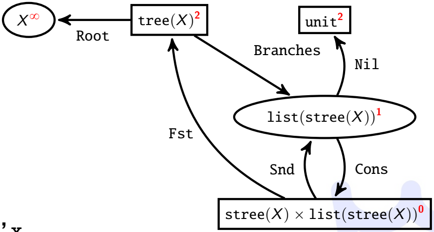


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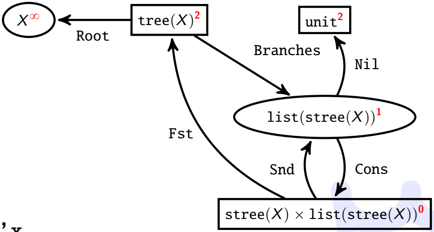


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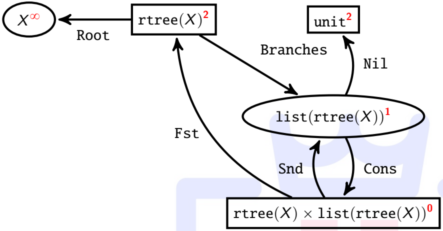
Theorem (L. Santocanale 2002)

Total elements of a type are exactly the winning strategies for the associated parity game.

Totality and strategies

Rules of the game:

- I play on odd vertices



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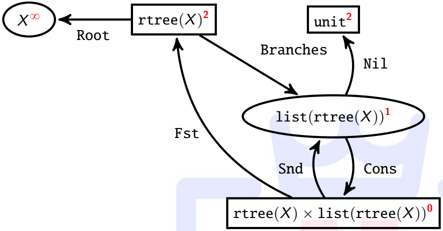

Totality and strategies

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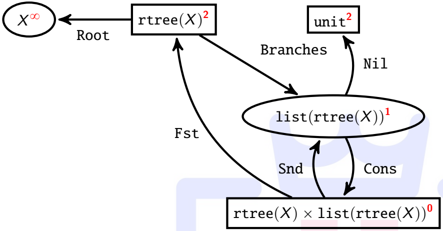
- I play on odd vertices
- I loose if I can't play
- if the play is infinite, I win if the maximum value that is visited infinitely often is even



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$$\texttt{Cons}^{1}\ \{\ \texttt{Fst}^{2} = \texttt{Succ}^{1}\ \texttt{x}\ ;\ \texttt{Snd}^{2} = \texttt{y}\ \}$$

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algorithm: SCP, yet again

What's missing

Some kind of definitions break the criterion:

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  idea: complement the criterion with "sized types", as in Agda.