Normalizing the Taylor expansion of (non-deterministic) λ-terms

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Böhm trees

- In the pure λ-calculus, we perform β-reduction

$$(\lambda x\, M)\ N \to_\beta M\,[N/x]$$

  contextually.
- By standardization, the leftmost outermost strategy reaches normal forms when they exist.
- Böhm trees = coinductive left normal forms

$$\mathsf{BT}\,(M) = \begin{cases} \lambda \overrightarrow{x}\ (y)\ \mathsf{BT}\,(N_1) \cdots \mathsf{BT}\,(N_k) & \text{if } M \simeq_\beta \lambda \overrightarrow{x}\ (y)\ N_1 \cdots N_k \\ \bot & \text{if } M \text{ has no hnf} \end{cases}$$

  e.g. $\mathsf{BT}\,(Y) = \lambda x\ (x)\ (x)\ (x)\ \cdots$
- Böhm trees form a syntactic denotational model, that is sensible

Note: here we do not consider extensionality issues

Infinitary normal forms in a non deterministic setting

Proposals for non deterministic Böhm trees exist:

- de'Liguoro and Piperno's Böhm trees for erratic choice (1995)
- recent work by Thomas Leventis on probabilistic Böhm trees

Alternative semantic approaches:

- in domain theory: powerdomains and the like (around 1980)
- Girard's quantitative semantics (around 1980)

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Here:

The "Böhm tree" of a non-deterministic λ-term is the normal form of its Taylor expansion

Quantitative semantics

A prime aged idea (Girard, '80s, before LL)

λ-terms = analytic functions = power series
Originally: for the λ-calculus, in an abstract categorical setting (coefficients are sets)

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Finiteness spaces (Ehrhard, early 2000’s)

Reformulate q.s. in linear logic using standard algebra:

- types $\rightsquigarrow$ particular topological vector spaces (or semimodules): $[\![A]\!] \subseteq \mathbf{S}^{|A|}$ + some additional structure
- function terms $\rightsquigarrow$ power series

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Differentiation of λ-terms (Ehrhard-Regnier 2003-2004)

So we can *differentiate* λ-terms, and compute their Taylor expansion!
And one can mimick that in the syntax:

- differential λ-calculus
- a finitary fragment: resource λ-calculus
  = the target of Taylor expansion

Resource λ-calculus

Resource terms

$$\begin{array}{rcl} \Delta & \ni & s, t, \ldots & ::= & x \mid \lambda x\, s \mid \langle s \rangle\ \bar{t} \\ !\Delta & \ni & \bar{s}, \bar{t}, \ldots & ::= & [s_1, \ldots, s_n] \end{array}$$

Resource reduction

$$\langle \lambda x\, s \rangle\ \bar{t} \to_\partial \partial_x s \cdot \bar{t} \quad \text{(anywhere)}$$

Semantically:

$$\partial_x s \cdot [s_1, \ldots, s_n] = \left(\tfrac{\partial^n s}{\partial x^n}\right)_{x=0} \cdot (s_1, \ldots, s_n)$$

Syntactically:

$$\partial_x s \cdot \bar{t} = \begin{cases} \sum_{f \in \mathfrak{S}_n} s\left[t_{f(1)}, \ldots, t_{f(n)}/x_1, \ldots, x_n\right] & \text{if } \mathbf{n}_x(s) = \#\bar{t} = n \\ 0 & \text{otherwise} \end{cases}$$

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- Linearity: $\lambda x\, 0 = 0$, $\langle s \rangle\ [t_1 + t_2, u] = \langle s \rangle\ [t_1, u] + \langle s \rangle\ [t_2, u]$, ...
- Resource reduction preserves free variables, is size-decreasing, strongly confluent and normalizing.

Taylor expansion of λ-terms

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Taylor expansion: $\Theta(M) \in \mathbf{S}^{\Delta}$

$$\Theta((M)\ N) = \sum_{n\in\mathbf{N}} \frac{1}{n!} \langle \Theta(M) \rangle\ \Theta(N)^n$$

$$\Theta(x) = x \qquad \Theta(\lambda x\, M) = \lambda x\, \Theta(M)$$

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Theorem (Ehrhard-Regnier, CiE 2006)

If $M \in \Lambda$, then $\Theta(M)$ normalizes to $\Theta(\mathsf{BT}(M))$.

Moral

In the ordinary λ-calculus $\mathsf{BT}(M) \simeq \mathsf{NF}(\Theta(M))$.

Normalizing Taylor expansions

But how can $\Theta(M)$ even normalize?

We want to set

$$\mathsf{NF}(\Theta(M)) = \sum_{s \in \Delta} \Theta(M)_s . \mathsf{NF}(s)$$

$\rightsquigarrow$ infinite sums *(and in general we might consider all kinds of coefficients)*
$\rightsquigarrow$ convergence?

Normalizing Taylor expansions: *uniformity to the rescue!*

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Theorem (Ehrhard-Regnier 2004, published in TCS in 2008)

Write $\mathcal{T}(M) = |\Theta(M)|$*. Then for all* $t \in \Delta$*, there is at most one* $s \in \mathcal{T}(M)$ *such that* $\mathsf{NF}(s)_t \neq 0$*.*

Proof.

λ-terms are uniform: their finitary approximants are pairwise coherent. □

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This fails in general

$\mathsf{NF}\left(\sum_{n \in \mathbf{N}} \langle \lambda x\, x \rangle^n [y]\right) = ?$ $\qquad$ $\langle \lambda x\, x \rangle^n [y] = \langle \lambda x\, x \rangle \left[\langle \lambda x\, x \rangle \left[\cdots [y] \cdots \right] \right]$

Taylor expansion in a non uniform setting

A generic quantitative non-uniform calculus

$$\Lambda_{\mathbf{S}} \ni M, N, \ldots ::= x \mid \lambda x\, M \mid (M)\ N \mid M + N$$

$$(\lambda x\, M)\ N \to_\beta M\,[N/x] \qquad (M + N)\ P = (M)\ P + (N)\ P$$

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$$(0)\; P = 0$$

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Example

Let $\delta_M = \lambda x\ (M + (x)\ x)$ *and* $\infty_M = (\delta_M)\ \delta_M$*:* $\infty_M \to_\beta{}^* M + \infty_M$.

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Then $\mathsf{NF}\,(\Theta\,(\infty_M)) = ?$
Worse: $\mathsf{NF}\,(\Theta\,(\infty_M - (\lambda x\, x)\ \infty_M)) = ?$

Finiteness structures to the rescue!

The main artifact of finiteness spaces:

Definition

- *If* $a, a' \subseteq A$*, write* $a \perp a'$ *iff* $a \cap a'$ *is finite.*
- *If* $\mathfrak{S} \subseteq \mathfrak{P}(A)$*, let* $\mathfrak{S}^{\perp} := \{a' \subseteq A\,;\forall a \in \mathfrak{S},\ a \perp a'\}$*.*
- *A finiteness structure is any* $\mathfrak{F} = \mathfrak{S}^{\perp}$*.*

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When is $\Theta(M)$ normalizable?

- Write $s \geq_\partial t$ if $s \to_\partial^* t + \cdots$.
- Let $\uparrow t = \{s \in \Delta \,;\, s \geq_\partial t\}$.
- $\Theta(M)$ is normalizable iff for all normal $t \in \Delta$, $\mathcal{T}(M) \perp \uparrow t$.
- $\{\uparrow t \,;\, t \text{ normal } \in \Delta\}^\perp$ is the finiteness structure of (supports of) normalizable vectors.

Typed terms have a finitary Taylor expansion

Let system F_+ be system F plus $\dfrac{\Gamma \vdash M : A \quad \Gamma \vdash N : A}{\Gamma \vdash M + N : A}$.

Theorem (Ehrhard, LICS 2010)

If $M \in \Lambda_{\mathbf{S}}$ is typable in system F_+, then $\mathcal{T}(M) \in \{\uparrow t\,;\, t \in \Delta\}^{\perp}$.

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Theorem (Pagani–Tasson–V., FoSSaCS 2016)

The same holds for all strongly normalizing terms, and we even have: $M \in \mathsf{SN}$ iff $\mathcal{T}(M) \in \{\uparrow a \; ; \, a \in \mathfrak{B}\}^{\perp}$ for a well chosen $\mathfrak{B}$.

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Theorem (Pagani–Tasson–V., *WIP*)

The above results generalize to (weak or head) normalizability.

In particular if M is normalizable then $\mathcal{T}(M) \in \{\uparrow s \,;\, s \text{ normal}\}^{\perp}$.

Conclusion

Conclusion (of my Jan. 2016 CHoCoLa talk)

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$$\mathsf{NF}\left(\Theta\left(M\right)\right) = \Theta\left(\mathsf{NF}\left(M\right)\right)$$

but the techniques of the uniform case are no longer available.

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  Theorem?
  If $M \to_\beta N$ then $\Theta(M) \rightsquigarrow \Theta(N)$.
- And then, what about non normalizing terms?

β-reduction through Taylor expansion

Recall that:

$$\Theta\left(\left(M\right)\ N\right) = \sum_{n \in \mathbf{N}} \frac{1}{n!} \left\langle \Theta\left(M\right)\right\rangle\ \Theta\left(N\right)^n$$

In quantitative semantics:

$$\llbracket(\lambda x\, M)\ N\rrbracket = \llbracket M\left[N/x\right]\rrbracket = \sum_{n \in \mathbf{N}} \frac{1}{n\,!} \left(\frac{\partial^n \llbracket M\rrbracket}{\partial x^n}\right)_{x=0} \cdot \llbracket N\rrbracket^n.$$

β-reduction through Taylor expansion: key steps

Promotion

$$\sigma^! := \sum_{n \in \mathbf{N}} \frac{1}{n\,!} \sigma^n \in \mathbf{S}^{!\Delta}$$

for all $\sigma \in \mathbf{S}^\Delta$

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$$\Theta\left(M\left[N/x\right]\right) = \partial_x \Theta\left(M\right) \cdot \Theta\left(N\right)^!$$

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Reduction

$$\langle \lambda x\, \sigma \rangle\ \tau^! \rightsquigarrow \partial_x \sigma \cdot \tau^!$$

and more generally

$$\langle \lambda x\, \sigma \rangle\ \overline{\tau} \rightsquigarrow \partial_x \sigma \cdot \overline{\tau} = \sum_{s\in\Delta, \overline{t}\in !\Delta} \sigma_s \overline{\tau}_{\overline{t}} . \partial_x s \cdot \overline{t}$$

β-reduction through Taylor expansion, step 1: problem

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Is this sum always defined?

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Is this sum always defined?
Yes (easy and already known): given $\bar{t} \in !\Delta$ and $n = \#\bar{t}$, there are finitely many $(t_1, \ldots, t_n)$ such that $\bar{t} = [t_1, \ldots, t_n]$.

β-reduction through Taylor expansion, step 2: problem

Substitution

$$\Theta\left(M\left[N/x\right]\right) = \partial_x \Theta\left(M\right) \cdot \Theta\left(N\right)^!$$

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Proof

By induction on M.

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In the application case, we need

$$\partial_x \sigma^! \cdot \tau^! = \left(\partial_x \sigma \cdot \tau^!\right)^! .$$

Syntactical version of the functoriality of exponentials in finiteness spaces, probabilistic coherence spaces, *etc.*

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Key lemma

$$\frac{\partial^k \left\langle s \right\rangle \overline{u}}{\partial x^k} \cdot \tau^k = \sum_{k_1 + k_2 = k} \frac{k\,!}{k_1\,!k_2\,!} \left\langle \frac{\partial^{k_1} s}{\partial x^{k_1}} \cdot \tau^{k_1} \right\rangle \frac{\partial^{k_2} \overline{u}}{\partial x^{k_2}} \cdot \tau^{k_2}$$

β-reduction through Taylor expansion, step 3: problem

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$$\langle \lambda x\, \sigma \rangle\ \overline{\tau} \rightsquigarrow \partial_x \sigma \cdot \overline{\tau} = \sum_{s \in \Delta, \overline{t} \in !\Delta} \sigma_s \overline{\tau}_{\overline{t}} . \partial_x s \cdot \overline{t}$$

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Is this sum always defined?
Yes (used by Ehrhard):

Lemma

If $s \succ_\partial t$ *(i.e.* $s \to_\partial t + \ldots$*) then* $\mathbf{s}\,(t) + 2 \leq \mathbf{s}\,(s) \leq 2\mathbf{s}\,(t) + 2$.

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Is this notion of reduction sufficient?
No: we need to reduce arbitrarily many redexes in parallel!

$$\Theta\left((y)\ (\lambda x\, x)\ z\right) = \sum_{n, k_1, \dots, k_n} \frac{1}{n! k_1! \cdots k_n!} \langle y \rangle\ \left[\langle \lambda x\, x \rangle\ z^{k_1}, \dots, \langle \lambda x\, x \rangle\ z^{k_n}\right]$$

$$\Theta\left((y)\ z\right) = \sum_{n \in \mathbf{N}} \frac{1}{n!} \langle y \rangle\ z^n$$

Parallel reduction on vectors of resource terms

Write $s \Rightarrow_\partial \sigma' \in \mathbf{N}[(!)\Delta]$ for the parallel reduction of resource terms.

Definition

Write $\sigma \mathrel{\widetilde{\Rightarrow_\partial}} \sigma'$ *if:*

$$\sigma = \textstyle\sum_{i \in I} a_i.s_i, \quad \sigma' = \textstyle\sum_{i \in I} a_i.\sigma'_i \quad \textit{and} \quad s_i \Rightarrow_\partial \sigma'_i \textit{ for all } i \in I.$$

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This is bad!

Consider $\sigma = \sum_{n\in\mathbf{N}} \langle \lambda x\, x \rangle^n z$

Parallel reduction on vectors of resource terms

Write $s \Rightarrow_\partial \sigma' \in \mathbf{N}[(!)\Delta]$ for the parallel reduction of resource terms.

Definition

Write $\sigma \widetilde{\Rightarrow_\partial} \sigma'$ if:

$$\sigma = \textstyle\sum_{i \in I} a_i.s_i, \quad \sigma' = \sum_{i \in I} a_i.\sigma'_i \quad \text{and} \quad s_i \Rightarrow_\partial \sigma'_i \text{ for all } i \in I.$$

This is bad!

Consider $\sigma = \sum_{n \in \mathbf{N}} \langle \lambda x\, x \rangle^n z$

Can we tame the combinatorial collapse of term size under parallel reduction?

Boundedly nested parallel reduction

Definition

Write $s \Rightarrow_{(b)} \sigma'$ *if* $s \Rightarrow_{\partial} \sigma'$ *so that the nesting tree of fired redexes is of height at most* b*.*

Lemma

For all $b \in \mathbf{N}$*, if* $s \Rightarrow_{(b)} s' + \ldots$ *then* $\mathbf{s}\,(s) \leq 4^b \mathbf{s}\,(s')$.

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$\bigcup_{b \in \mathbf{N}} \widetilde{\Rightarrow_{(\partial)}}$ b *is strongly confluent.*

Theorem

Restricted to vectors of bounded height, $\widetilde{\Rightarrow_{\partial}}$ *is strongly confluent.*

Normalizability

Standardization: normalizability = left-reduction to a normal form.

Inductive characterization

$$\frac{(M\,[N_0/x])\ N_1 \cdots N_n \Downarrow}{(\lambda x\, M)\ N_0\, N_1 \cdots N_n \Downarrow} \qquad \frac{N_1 \Downarrow \quad \cdots \quad N_n \Downarrow}{\lambda \overrightarrow{x}\ (y)\ N_1 \cdots N_n \Downarrow}$$

$$\frac{}{0 \Downarrow} \qquad \frac{M \Downarrow}{a.M \Downarrow} \qquad \frac{M \Downarrow \quad N \Downarrow}{M + N \Downarrow}$$

Note this is not stable under module equations: consider $\infty_M + (-1).\infty_M$.

Theorem

If $M \Downarrow$ then $\Theta\,(M)$ is normalizable and $\mathsf{NF}\,(\Theta\,(M)) = \Theta\,(\mathsf{NF}\,(M))$.

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Theorem

If $M \Downarrow$ *then* $\Theta\,(M)$ *is normalizable and* $\mathsf{NF}\,(\Theta\,(M)) = \Theta\,(\mathsf{NF}\,(M))$.

But we can do even better.

Unsolvable terms

Write $M \Uparrow$ if, for all $s \in |\Theta(M)|$, $\mathsf{NF}(s) = 0$.

Lemma

If $M \Uparrow$ *then* $\mathsf{NF}(\Theta(M)) = 0$.

Lemma

If $M \not\Uparrow$ *then* M *reduces to a sum containing a head normal form.*

Unsolvable terms *are not seen*.

Determinable terms

Definition (k-determinable terms)

$$\frac{}{M \Downarrow_0}\ (\Downarrow_0) \qquad \frac{M \Uparrow}{M \Downarrow_d}\ (\Uparrow) \qquad \frac{S \Downarrow_d}{\lambda x\, S \Downarrow_d}\ (\lambda) \qquad \frac{M_1 \Downarrow_d \quad \cdots \quad M_n \Downarrow_d}{(x)\ M_1 \cdots M_n \Downarrow_{d+1}}\ (v)$$

$$\frac{(S\,[M_0/x])\ M_1 \cdots M_n \Downarrow_d}{(\lambda x\, S)\ M_0 \cdots M_n \Downarrow_d}\ (r) \qquad \frac{M \Downarrow_d}{a.M \Downarrow_d}\ (\mathbf{S}) \qquad \frac{M \Downarrow_d \quad N \Downarrow_d}{M + N \Downarrow_d}\ (+)$$

- Write $M \Downarrow_\omega$ if $M \Downarrow_d$ for all d.
- Write $M\,\mathsf{df}_d$ if $M \Downarrow_d$ without (r).

Normalizing Taylor expansion, level by level

If $M \Downarrow_d$ then $M \to_\beta^* M'_d$ with $M'_d\,\mathsf{df}_d$.

Lemma

If $M \Downarrow_d$ then

- $\mathcal{T}(M) \in \{\uparrow s\ ;\ s \textit{ normal and of applicative depth } \leq d\}^\perp$
- *if s is in normal form and of applicative depth $\leq d$, then* $\mathsf{NF}(\Theta(M))_s = \Theta(M'_d)_s$ *for all* $M'_d \simeq_\beta M$ *with* $M'_d\,\mathsf{df}_d$

Theorem

If $M \Downarrow_\infty$ then $\mathcal{T}(M) \in \{\uparrow s\ ;\ s \textit{ normal}\}^\perp$ and $\mathsf{NF}(\Theta(M))_s = \Theta(\mathsf{NF}_d(M))_s$.

(where $\mathsf{NF}_d(M)$ is M'_d as above with all unsolvable subterms and all subterms at depth $\geq d$ replaced with 0)

- Generalizes Ehrhard-Regnier's result (with a new proof).
- Contains the case of normalizable algebraic λ-terms.

Conclusion

Normalization and Taylor expansion commute

provided it makes sense to normalize

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provided it makes sense to normalize

Further work

- give a precise account of standardization in a generic non-uniform setting (based on work by Thomas Leventis)
- adapt those results to proof nets (WIP with Jules Chouquet, Lorenzo Tortora, ...)
- we claim $\mathsf{BT}(M) = \mathsf{NF}(\Theta(M))$ when it is defined: does this coincide with existing notions of (non extensional) Böhm trees?
- when is Taylor expansion injective on normal forms? $\rightsquigarrow$ might lead to injectivity results for a class of quantitative denotational models
- generalization to infinitary λ-calculi?

The end

Questions?