Higher-order Arities, Signatures and Equations via Modules

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joint work with
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Keywords associated with syntax

Induction/Recursion

Substitution

Model

Syntax

Arity/Signature

Operation/Construction

This talk: give a *discipline* for specifying syntaxes

Motivating example: dLC

syntax of dLC $=$ ***differential λ-calculus*** [Ehrhard-Regnier 2003].

- explicitly involves **equations** e.g. $\mathrm{s}+\mathrm{t} = \mathrm{t}+\mathrm{s}$
- specifically taylored: (not an *instance* of a general framework/scheme)

inductive definition of a set + ad-hoc structure
e.g. **unary substitution**

Our proposal $=$ a discipline for presenting syntaxes

- signature = operations + equations
- [Fiore-Hure 2010]: alternative approach, for simply typed syntaxes

$\Rightarrow$ our approach explicitly relies on monads and modules (untyped case).

Syntax of dLC: [Ehrhard-Regnier 2003]

Let be given a denumerable set of variables. We define by induction on k an increasing family of sets (Δ_k). We set $\Delta_0 = \emptyset$ and Δ_{k+1} is defined as follows.

Monotonicity: if t belongs to Δ_k then t belongs to Δ_{k+1}.

Variable: if $n \in \mathbb{N}$, x is a variable, $i_1, \ldots, i_n \in \mathbb{N}^+ = \mathbb{N} \setminus \{0\}$ and $u_1, \ldots, u_n \in \Delta_k$, then

$$D_{i_1,\ldots,i_n} x \cdot (u_1, \ldots, u_n)$$

belongs to Δ_{k+1}. This term is identified with all the terms of the shape $D_{i_{\sigma(1)},\ldots,i_{\sigma(n)}} x \cdot (u_{\sigma(1)}, \ldots, u_{\sigma(n)}) \in \Delta_{k+1}$ where σ is a permutation on $\{1, \ldots, n\}$.

Abstraction: if $n \in \mathbb{N}$, x is a variable, $u_1, \ldots, u_n \in \Delta_k$ and $t \in \Delta_k$, then

$$D_1^n \lambda x\, t \cdot (u_1, \ldots, u_n)$$

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Application: if $s \in \Delta_k$ and $t \in R\langle \Delta_k \rangle$, then

$$(s)t$$

belongs to Δ_{k+1}.

Setting $n = 0$ in the first two clauses, and restricting application by the constraint that $t \in \Delta_k \subseteq R\langle \Delta_k \rangle$, one retrieves the usual definition of lambda-terms which shows that differential terms are a superset of ordinary lambda-terms.

The permutative identification mentioned above will be called *equality up to differential permutation*. We also work up to α-conversion.

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$(s)t \longleftarrow$ as an operation: $\Lambda \times \text{FreeCommutativeMonoid}(\Lambda) \to \Lambda$

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Syntax of dLC: [BEM 2010]

A **syntax** for the ***differential λ-calculus*** by ***mutual induction:***

[Bucciarelli-Ehrhard-Manzonetto 2010]

Simple terms:

$$\Lambda^s : \quad s, t \quad ::= \quad x \mid \lambda x.s \mid sT \mid \mathsf{D} s \cdot t$$

Differential λ-terms:

$$\Lambda^d : \quad T \quad ::= \quad 0 \mid s \mid s + T$$

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Syntax: specified by operations and equations.

But which ones are allowed ? What is the limit ?

Syntax of dLC: Our version

Which operations/equations are allowed to specify a syntax ?

A stand-alone presentation of differential λ-terms:

Allow sums everywhere (not only in the right arg of application)

Differential λ-terms:

$$\Lambda^{\mathrm{d}}: \quad S,T \quad ::= \quad x \mid \lambda x.S \mid S\,T \mid \mathsf{D}S \cdot T$$
$$\mid 0 \mid S + T$$

neutral element for $+$

modulo commutativity and associativity

Macros in [BEM 2010]:

$$\lambda x.\Sigma_i t_i := \Sigma_i \lambda x.t_i$$
$$(\Sigma_i t_i)u := \Sigma_i t_i u$$
$$D(\Sigma_i t_i) \cdot (\Sigma_j u_j) := \Sigma_i \Sigma_j D t_i \cdot u_j$$

Syntax of dLC: Conclusion

How can we compare these different versions ?

In which sense are they syntaxes ?

Which operations/equations are we allowed to specify in a syntax ?

Syntax of dLC: Conclusion

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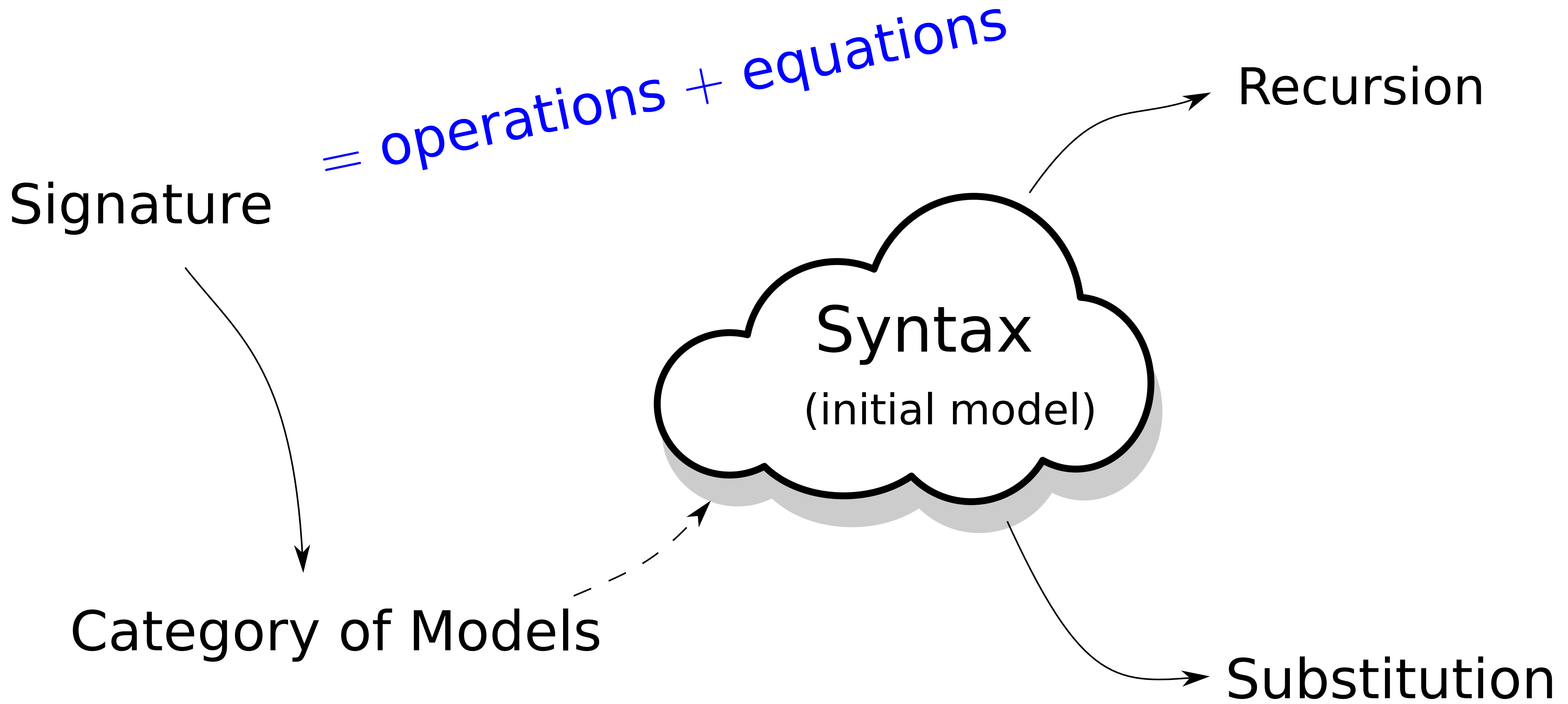
What is a syntax ?

What is a syntax?



generates a syntax = existence of the initial model

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Substitution and monads

Example: differential λ-calculus

$$\Lambda^{\mathrm{d}} : \quad S, T \quad ::= \quad x \mid \lambda x.\, S \mid S\,T \mid \mathrm{D}S \cdot T \mid 0 \mid S + T$$

Free variable indexing:

$$dLC : X \mapsto \{\text{terms taking free variables in } X\}$$

$$dLC(\emptyset) = \{0, \lambda z.z, \dots\}$$
$$dLC(\{x, y\}) = \{0, \lambda z.z, \dots, x, y, x + y, \dots\}$$

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Parallel substitution:

$$\mathrm{t} \quad \mapsto \quad \mathrm{t}[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})]$$

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Parallel substitution:

$$\begin{aligned} \mathrm{bind}_{\mathrm{f}} : dLC(\mathrm{X}) &\to dLC(\mathrm{Y}) \\ \mathrm{t} &\mapsto \mathrm{t}[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})] \end{aligned}$$

where $\mathrm{f} : \mathrm{X} \to dLC(\mathrm{Y})$

$\Rightarrow$ (dLC, var_X : X ⊂ dLC(X) , bind) $=$ **monad on** $\mathbf{Set}$

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$\Rightarrow$ (dLC, $\mathrm{var}_{\mathrm{X}}$: X ⊂ dLC(X) , bind) $=$ **monad on $\mathbf{Set}$**

monad morphism $=$ mapping preserving variables and substitutions.

Preview: Operations are module morphisms

+ commutes with substitution

$$(t+u)[x \mapsto v_x] = t[x \mapsto v_x] + u[x \mapsto v_x]$$

Categorical formulation

$dLC \times dLC$ supports dLC-substitution $\rightsquigarrow$ $dLC \times dLC$ is a **module over $\boldsymbol{dLC}$**

+ commutes with substitution $\rightsquigarrow$ $+ : dLC \times dLC \to dLC$ is a **module morphism**

Building blocks for specifying operations

Essential constructions of **modules over a monad** $\boldsymbol{R}$:

- R itself

- $M \times N$ for any modules M and N

  e.g. $\mathrm{R} \times \mathrm{R}$: $f : X \to R(Y)$

  $(\mathrm{t},\mathrm{u})[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})] := (\mathrm{t}[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})], \mathrm{u}[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})])$

- M' = **derivative of a module** $\boldsymbol{M}$: $M'(X) = M(X \coprod \{ \diamond \})$.

  disjoint union

  fresh variable

  used to model an operation binding a variable (Cf next slide).

Syntactic operations are module morphisms

operations = module morphisms = maps commuting with substitution.

$$0 : \quad 1 \quad \to \textcolor{red}{\mathrm{dLC}}$$

$$+ : \mathrm{dLC} \times \mathrm{dLC} \to \textcolor{red}{\mathrm{dLC}}$$

$$\mathrm{app} : \mathrm{dLC} \times \mathrm{dLC} \to \textcolor{red}{\mathrm{dLC}}$$

$$\mathrm{abs} : \mathrm{dLC}\textcolor{red}{'} \quad \to \textcolor{red}{\mathrm{dLC}}$$

$$\textcolor{blue}{\begin{aligned} \mathrm{abs}_X : \mathrm{dLC}(X \textstyle\coprod \{\diamond\}) &\to \mathrm{dLC}(X) \\ t &\mapsto \lambda\diamond.t \end{aligned}}$$

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Combining operations into a single one using disjoint union

$$[0, +] : 1 \coprod (\mathrm{dLC} \times \mathrm{dLC}) \to \textcolor{red}{\mathrm{dLC}}$$

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1-signatures and their models

A **1-signature** Σ = functorial assignment:

$$R \mapsto \Sigma(R)$$

Example: (0,+)

$$\Sigma_{0,+}(R) = 1 \coprod (R \times R)$$

A **model of Σ** is a pair:

$$(R, \quad \rho : \Sigma(R) \to R)$$

dLC $=$ model of $\Sigma_{0,+}$

$$[0,+] : 1 \coprod (dLC \times dLC) \to dLC$$

A **model morphism** m : (R,ρ) → (S,σ) $=$ monad morphism commuting with the module morphism:

$$\begin{tikzcd}
\Sigma(R) \arrow[r, "\rho"] \arrow[d, "\Sigma(m)"'] & R \arrow[d, "m"] \\
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Syntax

Definition

Given a 1-signature Σ, its **syntax** is an initial object in its category of models.

Question: Does the syntax exist for every 1-signature?

Answer: No.

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Counter-example: the 1-signature $R \mapsto \mathscr{P} \circ R$.

powerset endofunctor on Set

Examples of 1-signatures generating syntax

- **(0,+) language**:

  Signature: $R \mapsto 1 \coprod (R \times R)$

  Model: $(R\,,\quad 0 : 1 \to R\,,\quad + : R \times R \to R)$

  Syntax: $(B\,,\quad 0 : 1 \to B\,,\quad + : B \times B \to B)$

- **lambda calculus**:

  Signature: $R \mapsto R' \coprod (R \times R)$

  Model: $(R\,,\quad abs : R' \to R\,,\quad app : R \times R \to R)$

  Syntax: $(\Lambda\,,\quad abs : \Lambda' \to \Lambda\,,\quad app : \Lambda \times \Lambda \to \Lambda)$

Can we generalize this pattern?

Initial semantics for algebraic 1-signatures

Theorem [Hirschowitz & Maggesi 2007]

Syntax exists for any **algebraic 1-signature**, i.e. 1-signature built out of derivatives, products, disjoint unions, and the 1-signature $R \mapsto R$.

Algebraic 1-signatures correspond to the binding signatures described in [Fiore-Plotkin-Turi 1999]

(binding signature $=$ lists of natural numbers specify n-ary operations, possibly binding variables)

Question: Can we enforce some equations in the syntax ?

e.g. associativity and commutativity of $+$ for the differential λ-calculus.

Quotients of algebraic 1-signatures

[AHLM CSL 2018]: notion of ***quotients*** of 1-signatures.

Theorem [AHLM CSL 2018]

Syntax exists for any "***quotient***" **of algebraic 1-signature**.

Examples:

- a commutative binary operation
- application of the differential λ-calculus (original variant)

$$\text{app} : \text{dLC} \times \text{FreeCommutativeMonoid}(\text{dLC}) \to \text{dLC}$$

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... but not enough for the differential λ-calculus:

- associativity of $+$
- linearity of the operations

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Example: a commutative binary operation

Specification of a binary operation

1-Signature: $R \mapsto R \times R$

Model: $(R\,,\ + : R \times R \to R)$

What is an appropriate notion of model for a commutative binary operation ?

Example: a commutative binary operation

Specification of a commutative binary operation

1-Signature: $R \mapsto R \times R$

Model: $(R\,,\ + : R \times R \to R)$ s.t. $t + u = u + t$ (1)

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Answer: a monad equipped with a commutative binary operation

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Equation (1) states an equality between R-module morphisms:

$$
\begin{tikzcd}
\mathrm{R} \times \mathrm{R} \arrow[rr, bend left, "+"] \arrow[dr, "\text{swap}"'] & & \mathrm{R} \\
& \mathrm{R} \times \mathrm{R} \arrow[ur, "+"'] &
\end{tikzcd}
$$

Equations

Given a 1-signature Σ, (e.g. binary operation: $\Sigma(\mathrm{R}) = \mathrm{R} \times \mathrm{R}$)

a Σ-**equation** A $\rightrightarrows$ B is a functorial assignment:

$$R \mapsto \left(A(R) \rightrightarrows B(R) \right)$$

model of Σ

parallel pair of module morphisms over R

e.g. commutativity:

$$R \mapsto \left(R \times R \underset{+\circ swap}{\overset{+}{\rightrightarrows}} R \right)$$

A **2-signature** is a pair

$$(\Sigma, \mathrm{E})$$

1-signature

set of Σ-equations

***model* of a 2-signature** $(\boldsymbol{\Sigma}, \mathbf{E})$:

- a model R of Σ
- s.t. ∀ (A $\rightrightarrows$ B) $\in$ E, the two morphisms $A(R) \rightrightarrows B(R)$ are equal

Initial semantics for algebraic 2-signatures

Algebraic 2-signature: (Σ, E)

algebraic 1-signature

set of **elementary** Σ-equations

Theorem

Syntax exists for any algebraic 2-signature.

Main instances of **elementary** Σ-equations A $\rightrightarrows$ B:

- $A =$ algebraic 1-signature   e.g. A$(\mathrm{R}) = \mathrm{R} \times \mathrm{R}$
- $B(R) = R$

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- $B(R) = R$

Sketch of the construction of the syntax:

Quotient the initial model R of Σ by the following relation:

$\mathrm{x} \sim \mathrm{y}$ in $\mathrm{R}(\mathrm{X})$ iff for any model S of (Σ, E) , $\mathrm{i}(\mathrm{x}) = \mathrm{i}(\mathrm{y})$

initial Σ-model morphism $\mathrm{i} : \mathrm{R} \to \mathrm{S}$

Example: λ-calculus modulo βη

The algebraic 2-signature $(\Sigma_{LC\beta\eta}, E_{LC\beta\eta})$ of λ-calculus modulo βη:

$$\boldsymbol{\Sigma_{LC\beta\eta}}\,(R) := \Sigma_{LC}(R) = (R \times R) \coprod R'$$

model of $\boldsymbol{\Sigma_{LC}}$ = monad R with module morphisms:

$$\mathrm{app} : R \times R \to R \qquad \mathrm{abs} : R' \to R$$

β-equation: $(\lambda x.t)\ u = \underbrace{t[x \mapsto u]}_{\sigma_R(t,u)}$

η-equation: $t = \lambda x.(t\ x)$

$$\mathbf{E_{LC\beta\eta}} = \{\ \beta\text{-equation},\ \eta\text{-equation}\ \}$$

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The algebraic 2-signature $(\Sigma_{\mathrm{LC\beta\eta}}, E_{\mathrm{LC\beta\eta}})$ of λ-calculus modulo βη:

$$\boldsymbol{\Sigma}_{\mathbf{LC\beta\eta}}\,(\mathrm{R}) := \Sigma_{\mathrm{LC}}(\mathrm{R}) = (\mathrm{R} \times \mathrm{R}) \coprod \mathrm{R}'$$

model of $\boldsymbol{\Sigma}_{\mathbf{LC}}$ = monad R with module morphisms:

$$\mathrm{app} : \mathrm{R} \times \mathrm{R} \to \mathrm{R} \qquad \mathrm{abs} : \mathrm{R}' \to \mathrm{R}$$

β-equation: $(\lambda x.t)\ u = \underbrace{t[x \mapsto u]}_{\sigma_{\mathrm{R}}(t,u)}$

$$\begin{tikzcd} R' \times R \arrow[rr, bend left, "\sigma_{\mathrm{R}}"] \arrow[r, "\mathrm{abs} \times R"'] & R \times R \arrow[r, "\mathrm{app}"'] & R \end{tikzcd}$$

η-equation: $t = \lambda x.(t\ x)$

$$\begin{tikzcd} R \arrow[rr, bend left, "\mathrm{id}_{\mathrm{R}}"] \arrow[r, "R\iota_1"'] & R' \arrow[r, "\mathrm{abs}"'] & R \end{tikzcd}$$

$$\mathbf{E}_{\mathbf{LC\beta\eta}} = \{\ \beta\text{-equation},\ \eta\text{-equation}\ \}$$

Example: fixpoint operator

Definition [AHLM CSL 2018]

A **fixpoint operator** in a monad R is a module morphism $\mathrm{fix}\colon R' \to R$

s.t. for any term $t \in R(X \coprod \{\diamond\})$, $\mathrm{fix}(t) = t[\diamond \mapsto f(t)]$

Intuition:

- $\texttt{fix}(t) \ := \ \texttt{let rec} \diamond \texttt{ = t in } \diamond$
- [AHLM CSL 2018] Fixpoint operator in $LC_{\beta\eta} \simeq$ fixpoint combinators

Example: fixpoint operator

Definition [AHLM CSL 2018]

A **fixpoint operator** in a monad R is a module morphism $\mathrm{fix}\colon \mathrm{R}' \to \mathrm{R}$
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- [AHLM CSL 2018] Fixpoint operator in $\mathrm{LC}_{\beta\eta} \simeq$ fixpoint combinators

Algebraic 2-signature $(\Sigma_{\mathrm{fix}}, \mathrm{E}_{\mathrm{fix}})$ of a fixpoint operator:

$$\Sigma_{\mathrm{fix}}(\mathrm{R}) := \mathrm{R}'$$

$$E_{\mathtt{fix}} = \left\{ \begin{tikzcd} \underset{t}{R'} \arrow[r, bend left, "\mathtt{fix}(t)"] \arrow[r, bend right, "{t[\diamond \mapsto \mathtt{fix}(t)]}"'] & R \end{tikzcd} \right\}$$

Combining algebraic 2-signatures

Algebraic 2-signatures can be combined:

fixpoint operator

λ-calculus modulo βη

$$(\Sigma_{\mathrm{fix}}, E_{\mathrm{fix}}) \quad + \quad (\Sigma_{\mathrm{LC}\beta\eta}, E_{\mathrm{LC}\beta\eta})$$

$$=$$

$$(\Sigma_{\mathrm{fix}} \coprod \Sigma_{\mathrm{LC}\beta\eta} \;,\; E_{\mathrm{fix}} \cup E_{\mathrm{LC}\beta\eta})$$

λ-calculus modulo βη with an explicit fixpoint operator

Example: free commutative monoid

Algebraic 2-signature $(\Sigma_{\text{mon}}, \mathrm{E}_{\text{mon}})$ for the free commutative monoid monad:

$$\boldsymbol{\Sigma}_{\text{mon}}(\mathrm{R}) := 1 \coprod (\mathrm{R} \times \mathrm{R})$$

model of Σ_{mon} = monad R with module morphisms:

$$0 : 1 \to \mathrm{R} \qquad + : \mathrm{R} \times \mathrm{R} \to \mathrm{R}$$

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3 elementary Σ-equations:

$$\begin{tikzcd} \underset{s,t,u}{R \times R \times R} \arrow[r, bend left, "(s+t)+u"] \arrow[r, bend right, "s+(t+u)"'] & R \end{tikzcd} \qquad \begin{tikzcd} \underset{t}{R} \arrow[r, bend left, "0+t"] \arrow[r, bend right, "t"'] & R \end{tikzcd}$$

$$\begin{tikzcd} \underset{s,t}{R \times R} \arrow[r, bend left, "s+t"] \arrow[r, bend right, "t+s"'] & R \end{tikzcd}$$

Our target: dLC

Syntax of the *differential λ-calculus*:

Differential λ-terms

$$\begin{array}{rcll} s,t & ::= & \mathrm{x} & \\ & | & \lambda \mathrm{x.t} & \multirow{2}{*}{$\left.\vphantom{\begin{array}{c}a\\b\end{array}}\right\}$ λ-calculus} \\ & | & \mathrm{s\ t} & \\ & | & \mathrm{Ds} \cdot \mathrm{t} & \\ & | & \mathrm{s + t} & \\ & | & 0 & \end{array}$$

$\left.\begin{array}{l} \lambda \mathrm{x.t} \\ \mathrm{s\ t} \end{array}\right\}$ λ-calculus

$\left.\begin{array}{l} \mathrm{s + t} \\ 0 \end{array}\right\}$ free commutative monoid

and (bi)linearity of operations with respect to +:

$$\lambda \mathrm{x.(s+t)} = \lambda \mathrm{x.s} + \lambda \mathrm{x.t} \qquad \ldots$$

Algebraic 1-signature for dLC

Syntax of the *differential λ-calculus*:

Differential λ-terms — Corresponding 1-signature

$$
\begin{array}{rcll}
s,t & ::= & \mathrm{x} & (\text{variables} \subset \mathrm{R} \text{ for any monad } \mathrm{R}) \\
 & | & \lambda \mathrm{x}.\mathrm{t} & \multirow{2}{*}{$\left.\right\} \quad \Sigma_{\mathrm{LC}}(\mathrm{R}) = \mathrm{R}' \coprod (\mathrm{R} \times \mathrm{R})$} \\
 & | & \mathrm{s}\ \mathrm{t} & \\
 & | & \mathrm{Ds} \cdot \mathrm{t} & \mathrm{R} \mapsto \mathrm{R} \times \mathrm{R} \\
 & | & \mathrm{s} + \mathrm{t} & \multirow{2}{*}{$\left.\right\} \quad \Sigma_{\mathrm{mon}}(\mathrm{R}) = 1 \coprod (\mathrm{R} \times \mathrm{R})$} \\
 & | & 0 &
\end{array}
$$

Algebraic 1-signature for dLC

Syntax of the *differential λ-calculus*:

Differential λ-terms	Corresponding 1-signature
$s,t \ ::= \ \mathrm{x}$	$(\text{variables} \subset \mathrm{R} \text{ for any monad } \mathrm{R})$
$\mid \ \lambda\mathrm{x.t}$	$\Sigma_{\mathrm{LC}}(\mathrm{R}) = \mathrm{R}' \coprod (\mathrm{R} \times \mathrm{R})$
$\mid \ \mathrm{s\ t}$	
$\mid \ \mathrm{Ds} \cdot \mathrm{t}$	$\mathrm{R} \mapsto \mathrm{R} \times \mathrm{R}$
$\mid \ \mathrm{s} + \mathrm{t}$	$\Sigma_{\mathrm{mon}}(\mathrm{R}) = 1 \coprod (\mathrm{R} \times \mathrm{R})$
$\mid \ 0$	

Resulting algebraic 1-signature: $\Sigma_{\mathrm{dLC}}(\mathrm{R}) = \Sigma_{\mathrm{LC}}(\mathrm{R}) \coprod (\mathrm{R} \times \mathrm{R}) \coprod \Sigma_{\mathrm{mon}}(\mathrm{R})$

Elementary equations for dLC

Commutative monoidal structure:

$$\mathrm{E}_{\mathrm{mon}} \left\{ \begin{array}{c} \mathrm{s} + \mathrm{t} = \mathrm{t} + \mathrm{s} \\ \mathrm{s} + (\mathrm{t} + \mathrm{u}) = (\mathrm{s} + \mathrm{t}) + \mathrm{u} \\ 0 + \mathrm{t} = \mathrm{t} \end{array} \right. \qquad \begin{array}{r} \mathrm{R}\times\mathrm{R} \rightrightarrows \mathrm{R} \\ \mathrm{R}\times\mathrm{R}\times\mathrm{R} \rightrightarrows \mathrm{R} \\ \mathrm{R} \rightrightarrows \mathrm{R} \end{array}$$

Linearity:

$$\begin{array}{cr} \lambda\mathrm{x}.(\mathrm{s}+\mathrm{t}) = \lambda\mathrm{x}.\mathrm{s} + \lambda\mathrm{x}.\mathrm{t} & \mathrm{R}\times\mathrm{R} \rightrightarrows \mathrm{R} \\ \mathrm{D}(\mathrm{s}+\mathrm{t})\cdot\mathrm{u} = \mathrm{D}\mathrm{s}\cdot\mathrm{u} + \mathrm{D}\mathrm{t}\cdot\mathrm{u} & \mathrm{R}\times\mathrm{R}\times\mathrm{R} \rightrightarrows \mathrm{R} \\ \mathrm{D}\mathrm{s}\cdot(\mathrm{t}+\mathrm{u}) = \mathrm{D}\mathrm{s}\cdot\mathrm{t} + \mathrm{D}\mathrm{s}\cdot\mathrm{u} & \mathrm{R}\times\mathrm{R}\times\mathrm{R} \rightrightarrows \mathrm{R} \end{array}$$

$$\cdots$$

n-ary fixpoint operator

Reminder: unary fixpoint operator in a monad R

$$\begin{array}{ccc} \mathrm{R}(\mathrm{X} \coprod \{\diamond\}) & \to & \mathrm{R}(\mathrm{X}) \\ t & \mapsto & \bar{t} \end{array} \qquad \textbf{s.t.} \qquad t[\diamond \mapsto \bar{t}] = \bar{t}$$

Intuition: $\bar{t} \quad := \quad$ `let rec ◇ = t in ◇`

n-ary fixpoint operator:

$$\forall\, i \in \{1,..,n\}, \quad \begin{array}{ccc} \mathrm{R}(\mathrm{X} \coprod \{\diamond_1, \ldots, \diamond_n\})^{n} & \to & \mathrm{R}(\mathrm{X}) \\ t_1, \ldots, t_n & \mapsto & \overline{t_i} \end{array} \qquad \textbf{s.t.} \quad \forall\, i,\ t_i \begin{bmatrix} \diamond_1 \mapsto \overline{t_1} \\ \cdots \\ \diamond_n \mapsto \overline{t_n} \end{bmatrix} = \overline{t_i}$$

Intuition: $\overline{t_i} \quad := \quad$ `let rec ◇₁ = t₁ and .. and ◇ₙ = tₙ in ◇ᵢ`

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n-ary fixpoint operator:

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Intuition: $\overline{t_i} \quad := \quad$ `let rec ◇₁ = t₁ and .. and ◇ₙ = tₙ in ◇ᵢ`

$\Rightarrow$ specifiable as an algebraic 2-signature

Fixpoint operators

Syntax with fixpoint operators:

- for each n, a n-ary operator:

  $\texttt{let rec } \diamond_1 = t_1 \texttt{ and .. and } \diamond_n = t_n \texttt{ in } \diamond_i$

- compatibility between these operators [AHLM CSL 2018]

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  - invariance under **permutation**:

$$\begin{array}{l} \mathtt{let\ rec}\ \diamond_1 = t_1 \\ \quad \mathtt{and}\ \diamond_2 = t_2 \\ \mathtt{in}\ \diamond_1 \end{array} \quad = \quad \begin{array}{l} \mathtt{let\ rec}\ \diamond_1 = t_2[\diamond_1 \leftrightarrow \diamond_2] \\ \quad \mathtt{and}\ \diamond_2 = t_1[\diamond_1 \leftrightarrow \diamond_2] \\ \mathtt{in}\ \diamond_2 \end{array}$$

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  - invariance under **repetition**:

$$\begin{array}{l} \texttt{let rec } \diamond_1 = t \\ \qquad \texttt{and } \diamond_2 = t \\ \texttt{in } \diamond_1 \end{array} \quad = \quad \begin{array}{l} \texttt{let rec } \diamond_1 = t[\diamond_2 \mapsto \diamond_1] \\ \texttt{in } \diamond_1 \end{array}$$

Fixpoint operators

Syntax with fixpoint operators:

- for each n, a n-ary operator:

$$\texttt{let rec } \diamond_1 = t_1 \texttt{ and } .. \texttt{ and } \diamond_n = t_n \texttt{ in } \diamond_i$$

- compatibility between these operators [AHLM CSL 2018]

general form:

$$\boxed{\begin{array}{l} \texttt{let rec } \diamond_1 = t_1[\diamond_i \mapsto \diamond_{u(i)}] \\ \qquad \dots \\ \qquad \texttt{and } \diamond_q = t_q[\diamond_i \mapsto \diamond_{u(i)}] \\ \texttt{in } \diamond_{u(j)} \end{array}} = \boxed{\begin{array}{l} \texttt{let rec } \diamond_1 = t_{u(1)} \\ \qquad \dots \\ \qquad \texttt{and } \diamond_p = t_{u(p)} \\ \texttt{in } \diamond_j \end{array}}$$

where $u : \{1, \dots, p\} \to \{1, \dots, q\}$

$$t_1, \dots, t_q \in R(X \coprod \{\diamond_1, \dots, \diamond_p\})$$

Fixpoint operators

Syntax with fixpoint operators:

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  ```
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$\Rightarrow$ Expressible as elementary equations $(\mathrm{R}^{,\dots,})^{\mathrm{q}} \rightrightarrows \mathrm{R}$.

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Principle of recursion

Recursion on the syntax ≃ Initiality in the category of models

Recipe for constructing "by recursion" a monad morphism:

$$\mathrm{f} : \mathrm{R} \to \mathrm{S}$$

initial model of a 2-signature (Σ, E)

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Initiality of R $\Rightarrow$ model morphism $\mathrm{R} \to \mathrm{S}$ $\Rightarrow$ monad morphism $\mathrm{R} \to \mathrm{S}$

Example: Computing the set of free variables

λ-calculus monad

power set monad

$$\begin{aligned} \mathrm{fv}_X \quad : \quad \mathrm{LC}(X) & \to \mathcal{P}(X) \\ t & \mapsto \text{(exact) set of free variables of } t \end{aligned}$$

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LC = initial model of $(\Sigma_{\mathrm{LC}}\,,\varnothing)$

$\Sigma_{\mathrm{LC}}(\mathrm{R}) = (\mathrm{R} \times \mathrm{R}) \coprod \mathrm{R}'$

$\Rightarrow$ make $\mathcal{P}$ a model of Σ_{LC}

$\cup :\; \mathcal{P} \times \mathcal{P} \to \mathcal{P}$

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Initiality of $\mathrm{LC} \;\Rightarrow\; \mathrm{fv} : \mathrm{LC} \to \mathcal{P}$

Equalities as a monad morphism:

$$\mathrm{fv}(x) = \{x\} \qquad\qquad \mathrm{fv}(t[x \mapsto u(x)]) = \bigcup_{x \in \mathrm{fv}(t)} \mathrm{fv}(u(x))$$

Equalities as a model morphism:

$$\mathrm{fv(app(t,u))} = \mathrm{fv(t)} \cup \mathrm{fv(u)} \qquad\qquad \mathrm{fv}(\mathrm{abs}(t)) = \mathrm{fv}(t) \setminus \{\diamond\}$$

Example: Translating λ-calculus with fixpoint

λ-calculus modulo βη
+ fixpoint operator fix

compilation

$$\Longrightarrow$$

$$\mathrm{fix}(t) \quad \mapsto \quad ?$$

λ-calculus modulo βη

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...as a monad morphism $\mathrm{LC}_{\beta\eta+\mathrm{fix}} \to \mathrm{LC}_{\beta\eta}$

$\mathrm{LC}_{\beta\eta+\mathrm{fix}}$ = initial model of $(\Sigma_{\mathrm{LC}\beta\eta}, \mathrm{E}_{\mathrm{LC}\beta\eta}) + (\Sigma_{\mathrm{fix}}, \mathrm{E}_{\mathrm{fix}})$

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Proposition [AHLM CSL 2018]

Fixpoint operators in $\mathrm{LC}_{\beta\eta}$ are in one to one correspondance with fixpoint combinators (i.e. λ-terms Y s.t. $t\,(Y\,t) = Y\,t$ for any t).

Example: Translating λ-calculus with fixpoint

λ-calculus modulo βη
+ fixpoint operator fix

compilation

$\Longrightarrow$

λ-calculus modulo βη

$\mathrm{fix}(\mathrm{t}) \;\mapsto\; \mathrm{app}(\mathrm{Y}, \mathrm{abs}(\mathrm{t}))$

a chosen fixpoint combinator

...as a monad morphism $\mathrm{LC}_{\beta\eta+\mathrm{fix}} \to \mathrm{LC}_{\beta\eta}$

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$$\mathrm{fix}(t) \;\mapsto\; \mathrm{app}(Y, \mathrm{abs}(t))$$

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...as a monad morphism $LC_{\beta\eta+\mathrm{fix}} \to LC_{\beta\eta}$

$LC_{\beta\eta+\mathrm{fix}}$ = initial model of $(\Sigma_{LC\beta\eta}, E_{LC\beta\eta}) + (\Sigma_{\mathrm{fix}}, E_{\mathrm{fix}})$

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$\hat{Y} : t \mapsto \mathrm{app}(Y, \mathrm{abs}(t))$

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Fixpoint operators in $LC_{\beta\eta}$ are in one to one correspondance with fixpoint combinators (i.e. λ-terms Y s.t. $t\,(Y\,t) = Y\,t$ for any t).

Initiality of $LC_{\beta\eta+\mathrm{fix}} \Rightarrow$ monad morphism $LC_{\beta\eta+\mathrm{fix}} \to LC_{\beta\eta}$

Example: Computing the size of a term

λ-calculus monad

$$\begin{aligned} \mathrm{s}_{\mathrm{X}} \;:\; \mathrm{LC}(\mathrm{X}) &\to \mathbb{N} \\ t &\mapsto \text{number of constructors in } t \end{aligned}$$

$$s(x) = 0$$
$$s(\lambda x.x) = 1$$
$$s((\lambda x.x)\ y) = 2$$

.. as a monad morphism $\mathrm{s} : \mathrm{LC} \to \mathbb{N}$

Example: Computing the size of a term

λ-calculus monad

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$$s(x) = 0$$
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Intuition: $f_X : \mathrm{LC}(X) \to \mathbb{N}^{(\mathbb{N}^X)} \quad \overset{\textit{uncurry}}{\Longrightarrow} \quad g : \mathrm{LC}(X) \times \mathbb{N}^X \to \mathbb{N}$

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$$\mathrm{s(t) = g(t,\ (x \mapsto 0))}$$

variables are of size 0

Conclusion

Summary of the talk:

- notion of 1-signature and models based on monads and modules
- 2-signature = 1-signature + set of equations
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Thank you!