Sharing Equality is Linear

Andrea Condoluci

Joint work with Beniamino Accattoli
Claudio Sacerdoti Coen

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ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

inventeurs du monde numérique

Structure of the Presentation

Evaluation & Conversion
- Complexity
- Sharing

Related Works

The Theory of Sharing Equality
- λ-Graphs
- Queries
- Sharing Equivalences

Linear-Time Algorithm
- First-order Check
- Variables Check

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Evaluation & Conversion

- Fix a dialect λ_X of the λ-calculus
- With a deterministic evaluation strategy $\rightarrow_X$
- $\mathtt{nf}_X(t)$ is the normal form of t with respect to $\rightarrow_X$

Evaluation

Given t, computing $\mathtt{nf}_X(t)$

Conversion

Given t and u, checking whether $\mathtt{nf}_X(t) =_\alpha \mathtt{nf}_X(u)$

Complexity

What is the complexity

of evaluation and conversion?

Parameters

- Input term: size of the initial term $|t|$
- Number of steps: number n such that $t \to_X^n \mathtt{nf}_X(t)$

Size explosion 💥

There exists a family $\{t_n\}_{n\in\mathbb{N}}$ such that (for all λ_X):

$$t_n \ \to_X^n \ \mathtt{nf}_X(t_n)$$

with

$$|t_n| \in O(n) \ \text{ and } \ |\mathtt{nf}_X(t_n)| \in \Omega(2^n)$$

Consequences

- Evaluation is exponential in n and $|t|$
- Conversion is also exponential

Sharing is caring ♥

Add sharing to λ_X, obtaining $\lambda_{\mathtt{sh}X}$:

$$\begin{tikzcd} \lambda_X \arrow[rr, dotted] \arrow[dr] & & \mathrm{RAM} \\ & \lambda_{\mathtt{sh}X} \arrow[ur] & \end{tikzcd}$$

1. Turn to shared evaluation $\to_{\mathtt{sh}X}$
2. Compute shared normal forms $\mathtt{nf}_{\mathtt{sh}X}(t)$
3. Simulating $\to_X$ up to sharing unfolding

   i.e. so that $\mathtt{nf}_{\mathtt{sh}X}(t)\downarrow = \mathtt{nf}_X(t)$

Evaluation w/ sharing

Compute $\mathtt{nf}_{\mathtt{sh}X}(t)$ instead of $\mathtt{nf}_X(t)$

Call-by-Value (CbV)

Let $t \to_{CbV}^{n} \mathtt{nf}_{CbV}(t)$

- Blelloch & Greiner, 1995

  Polynomial in $|t|$ and n

  ⋮

- Accattoli & Condoluci & Sacerdoti Coen, 2019

  Linear in $|t|$ and n

Conversion w/ Sharing

Given t and u:

1. Evaluation: computing $\mathtt{nf}_{\mathtt{sh}X}(t)$ and $\mathtt{nf}_{\mathtt{sh}X}(u)$
2. Sharing equality: checking $\mathtt{nf}_{\mathtt{sh}X}(t)\!\downarrow\, =_\alpha \mathtt{nf}_{\mathtt{sh}X}(u)\!\downarrow$

Evaluation is bilinear... what about sharing equality?

Problems

- It can be $t \neq_\alpha u$ and yet $t\!\downarrow\, =_\alpha u\!\downarrow$
- Sharing unfolding is exponential

Polynomial sharing equality

→ Testing without unfolding

Sharing Equality is Linear

Sharing equality

Given shared t and u, checking $t{\downarrow} =_\alpha u{\downarrow}$

- Accatoli & Dal Lago, 2012

  Sharing equality is $O((|t| + |u|)^2)$

  ➡ Conversion is biquadratic — thus reasonable

- This talk: sharing equality is $O(|t| + |u|)$

  ➡ Conversion is bilinear

Structure of the Presentation

Algorithms for Sharing Equality

Sharing equality

Given shared t and u, checking $t{\downarrow} =_\alpha u{\downarrow}$

Let $n := |t| + |u|$

Existing algorithms

- Accattoli & Dal Lago, 2012
  $O(n^2)$ algorithm based on dynamic programming
- Grabmayer & Rochel, 2014
  $O(n \log n)$ algorithm for λ-terms with `letrec`
  (more general problem)

Related Problems

- First-order unification

  Martelli & Montanari, 1977; Paterson & Wegman, 1978

- Nominal unification:

  Two algorithms, adapting MM and PW, quadratic

  Calvès & Fernandez and Levy & Villaret, 2010-13

- Nominal matching: linear only on unshared terms

  Calvès & Fernandez 2010

- Pattern unification:

  PW based, claimed linear — seems quadratic

  Qian 1993

- DFAs equivalence: pseudo-linear

  Hopcroft & Karp 1971

This Talk

Andrea Condoluci, Beniamino Accattoli and Claudio Sacerdoti Coen, Sharing Equality is Linear, PPDP 2019, October 7^{th}, 2019 Porto, Portugal.

https://arxiv.org/abs/1907.06101

- A simple theory of sharing equality for λ-terms as DAGs $\Rightarrow$ λ-graphs
- A linear-time, two-phases algorithm
  1. First-order check: based on PW
  2. Variables check: binders and variables are shared correctly

  The splitting in two steps comes from the developed theory

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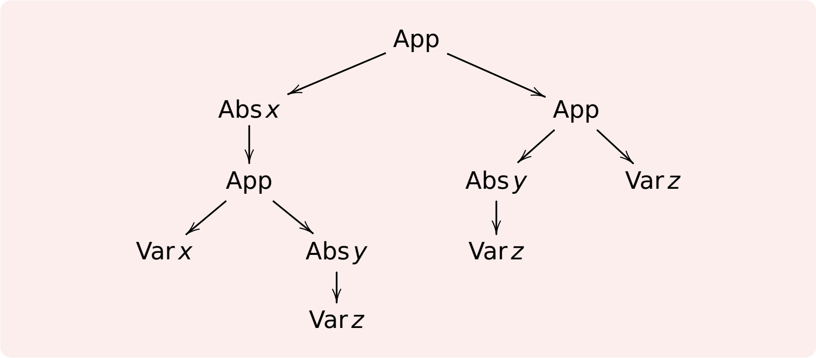
Linear-Time Algorithm
- First-order Check
- Variables Check

Syntax Tree

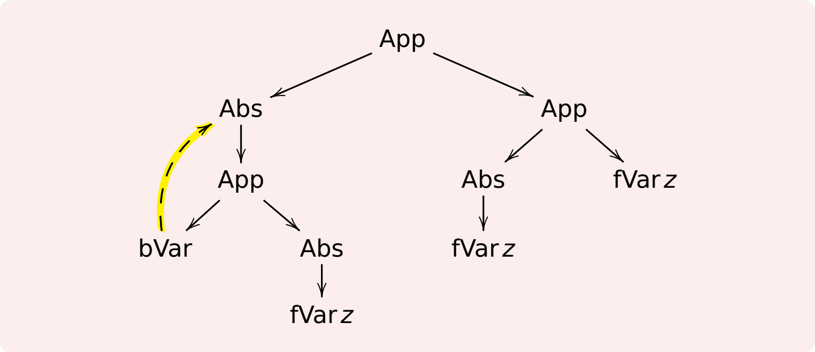
$$(\lambda x.\, x\,(\lambda y.z))\,((\lambda y.z)\,z)$$



Three kinds of nodes: App, Abs and Var

Syntax Tree (ii)

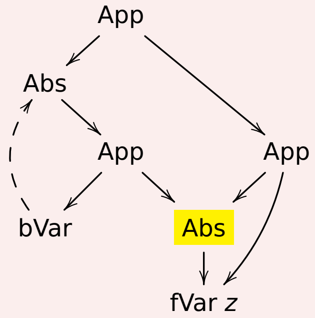
$$(\lambda x.\, x\,(\lambda y.z))\,((\lambda y.z)\,z)$$



Bound variables (bVar) have a binding edge — the dashed one

λ-graph

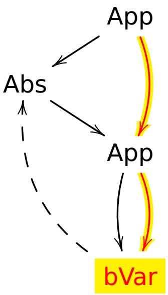
$$(\lambda x.\, x\ \boxed{\lambda y.z}\,)\,(\,\boxed{\lambda y.z}\ z)$$



Sharing in-degree > 1 (excluding binding edges)

Note: sharing of abstractions and under abstractions

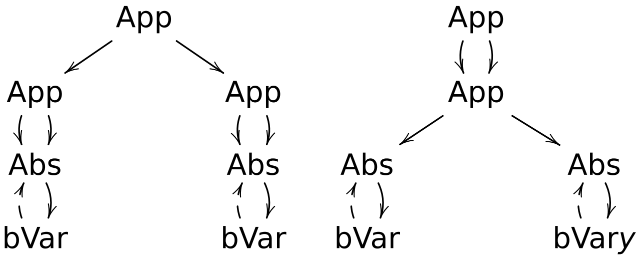
Structural Conditions



¿ $(\lambda x.xx)$ xx ?

- DAG: the graph is acyclic (excluding binding edges)
- Well-formed scopes: each λ-node dominates the bVar-nodes it binds

Sharing Equality



$$((\lambda x.x)\ (\lambda x.x))\ ((\lambda y.y)\ (\lambda y.y))$$

vs.

$$((\lambda x.x)\ (\lambda y.y))\ ((\lambda x.x)\ (\lambda y.y))$$

Problem

Are the two λ-graphs sharing equal?

➡ iff there exists a sharing equivalence

Bisimulations

Sharing equivalence is roughly a bisimulation

Definition (Bisimulation)

A binary relation $\mathcal{B}$ over the nodes of a λ-graph is a bisimulation if it is:

- Homogeneous: $\mathcal{B}$ relates only nodes of the same kind
- Compatible: $\mathcal{B}$ is closed under the following rules

$$\frac{\mathsf{App}(n_1, n_2)\ \mathcal{B}\ \mathsf{App}(m_1, m_2)}{n_1\ \mathcal{B}\ m_1}\ \boxed{\swarrow} \qquad \frac{\mathsf{App}(n_1, n_2)\ \mathcal{B}\ \mathsf{App}(m_1, m_2)}{n_2\ \mathcal{B}\ m_2}\ \boxed{\searrow}$$

$$\frac{\mathsf{Abs}(n)\ \mathcal{B}\ \mathsf{Abs}(m)}{n\ \mathcal{B}\ m}\ \boxed{\downarrow} \qquad \frac{\mathsf{bVar}(n)\ \mathcal{B}\ \mathsf{bVar}(m)}{n\ \mathcal{B}\ m}\ \boxed{\circlearrowright}$$

Sharing Equivalence

Definition (Sharing equivalence)

A binary relation $\equiv$ over the nodes of a λ-graph is a sharing equivalence if it is:

Equivalence $\equiv$ is an equivalence relation

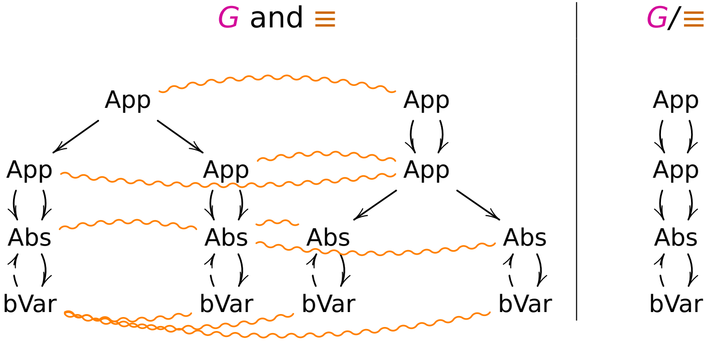
Bisimulation $\equiv$ is a bisimulation

Open if $\mathrm{fVar}(x) \equiv \mathrm{fVar}(y)$ then $x = y$

Sanity check

If G is a λ-graph and $\equiv$ is a sharing equivalence over G, then $G/\equiv$ is a λ-graph.

Example



G and ≡
App
App
App
App
Abs
Abs
Abs
Abs
bVar
bVar
bVar
bVar
G/≡
App
App
Abs
bVar

Sharing Equality Problem

Input A λ-graph G + two root nodes n and m of G

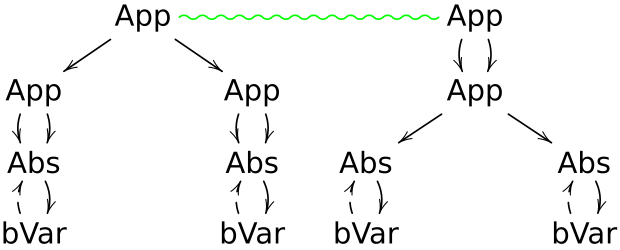
Problem Is there a sharing equivalence $\equiv$ on G such that $n \equiv m$?

More generally:

Input G + a query $\mathcal{Q}$ (any relation over roots)

Problem Is there a sharing equivalence $\equiv$ on G containing $\mathcal{Q}$?

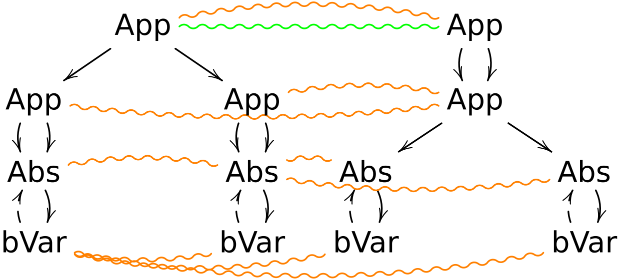
Example



Problem

Is there a sharing equivalence $\equiv$ containing a given query $\mathcal{Q}$?

Example



Problem

Is there a sharing equivalence $\equiv$ containing a given query $\mathcal{Q}$? Yep.

Propagated Queries

Propagation $(\cdot)\Downarrow$

Let $\mathcal{Q}$ be a query. Its propagation $\mathcal{Q}\Downarrow$ is the smallest relation closed under the following rules:

$$\swarrow\ \frac{\mathsf{App}(n_1, n_2)\ \mathcal{B}\ \mathsf{App}(m_1, m_2)}{n_1\ \mathcal{B}\ m_1} \qquad \searrow\ \frac{\mathsf{App}(n_1, n_2)\ \mathcal{B}\ \mathsf{App}(m_1, m_2)}{n_2\ \mathcal{B}\ m_2}$$

$$\downarrow\ \frac{\mathsf{Abs}(n)\ \mathcal{B}\ \mathsf{Abs}(m)}{n\ \mathcal{B}\ m} \qquad \circlearrowright\ \cancel{\frac{\mathsf{bVar}(n)\ \mathcal{B}\ \mathsf{bVar}(m)}{n\ \mathcal{B}\ m}}$$

Universality of $\mathcal{Q}\Downarrow$

If there is an open bisimulation containing $\mathcal{Q}$, then $\mathcal{Q}\Downarrow$ is the smallest open bisimulation containing $\mathcal{Q}$

Spreaded Queries

Spreading $(\cdot)\#$

Let $\mathcal{Q}$ be a query. Its spreading $\mathcal{Q}\#$ is the smallest equivalence relation closed under:

$$\swarrow \quad \frac{\mathsf{App}(n_1, n_2) \mathrel{\mathcal{B}} \mathsf{App}(m_1, m_2)}{n_1 \mathrel{\mathcal{B}} m_1} \qquad \searrow \quad \frac{\mathsf{App}(n_1, n_2) \mathrel{\mathcal{B}} \mathsf{App}(m_1, m_2)}{n_2 \mathrel{\mathcal{B}} m_2}$$

$$\downarrow \quad \frac{\mathsf{Abs}(n) \mathrel{\mathcal{B}} \mathsf{Abs}(m)}{n \mathrel{\mathcal{B}} m} \qquad \circlearrowleft \quad \xcancel{\frac{\mathsf{bVar}(n) \mathrel{\mathcal{B}} \mathsf{bVar}(m)}{n \mathrel{\mathcal{B}} m}}$$

Universality of $\mathcal{Q}\#$

If there exists a sharing equivalence containing $\mathcal{Q}$, then $\mathcal{Q}\#$ is the smallest sharing equivalence containing $\mathcal{Q}$

Sharing Equality Theorem

There exists a sharing equivalence containing $\mathcal{Q}$

$\Updownarrow$

$[\![\mathcal{Q}]\!]$ holds, *i.e.* $[\![n]\!] = [\![m]\!]$ for all $n \; \mathcal{Q} \; m$

Sharing Equality Theorem

There exists a sharing equivalence containing $\mathcal{Q}$

$\Updownarrow$

$\mathcal{Q}\#$ is a sharing equivalence

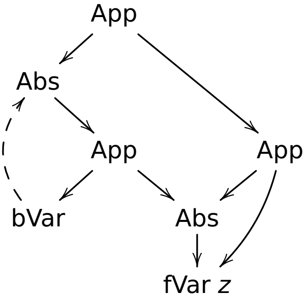
$\Updownarrow$

$\mathcal{Q}\Downarrow$ is an open bisimulation

$\Updownarrow$

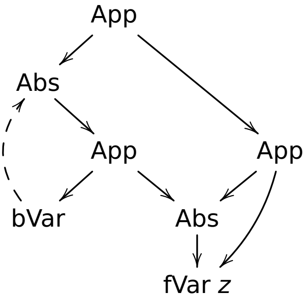
$[\![\mathcal{Q}]\!]$ holds, *i.e.* $[\![n]\!] = [\![m]\!]$ for all $n \; \mathcal{Q} \; m$

Read Back



$$
\begin{array}{rrcl}
\textit{Application}: & [\![\mathrm{App}(n,m)]\!] & := & [\![n]\!]\,[\![m]\!] \\
\textit{Abstraction}: & [\![\mathrm{Abs}(n)]\!] & := & \lambda ?.\ [\![n]\!] \\
\textit{Bound Variable}: & [\![\mathrm{bVar}(n)]\!] & := & ? \\
\textit{Free Variable}: & [\![\mathrm{fVar}(x)]\!] & := & x
\end{array}
$$

Read Back — Locally Nameless



- $[\![\tau : r \leadsto \mathrm{App}(n, m)]\!] \quad := \quad [\![(\tau\swarrow) : r \leadsto n]\!]\,[\![(\tau\searrow) : r \leadsto m]\!]$
- $[\![\tau : r \leadsto \mathrm{Abs}(n)]\!] \quad := \quad \lambda [\![(\tau\downarrow) : r \leadsto n]\!]$
- $[\![\tau : r \leadsto \mathrm{bVar}(n)]\!] \quad := \quad \underline{\mathrm{indexOf}(n \mid \tau : r)}$
- $[\![\tau : r \leadsto \mathrm{fVar}(x)]\!] \quad := \quad x$

$$[\![\epsilon : r \leadsto r]\!] = (\lambda\ \underline{0}\ (\lambda\ z))\ ((\lambda\ z)\ z)$$

Sharing Equality Theorem

There exists a sharing equivalence containing $\mathcal{Q}$

$\Updownarrow$

$\mathcal{Q}\#$ is a sharing equivalence

$\Updownarrow$

$\mathcal{Q}\Downarrow$ is an open bisimulation

$\Updownarrow$

$[\![\mathcal{Q}]\!]$ holds, *i.e.* $[\![n]\!] = [\![m]\!]$ for all $n \ \mathcal{Q} \ m$

Sharing Equality Theorem

There exists a sharing equivalence containing $\mathcal{Q}$

$$\Updownarrow$$

$\mathcal{Q}\#$ is a sharing equivalence

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The Two Phases

Checking sharing equality

Compute $\mathcal{Q}\#$, then check that it is a sharing equivalence

Two phases

1. First-order check: is $\mathcal{Q}\#$ a FO bisimulation ?
2. Variables check: check variables and scopes

Once the first phase is solved, the second one is straightforward

Note: the decomposition relies on the theory

Checking Sharing Equality

Compute $\mathcal{Q}\#$, then check that it is a sharing equivalence

Difficulty 1

- $\mathcal{Q}\#$ is an equivalence relation
- Equivalence relations have size quadratic in the no. of nodes
- Idea: use a canonic element-based representation of $\mathcal{Q}\#$

Difficulty 2

- Linearity requires to never merge equivalence classes
- Idea (naïve): propagating $\mathcal{Q}$ downward by levels

Paterson & Wegman

Difficulty 2: linearity requires to never merge equivalence classes

Paterson & Wegman idea

1. Start wherever
2. To process each node:
   2.1 Process first the parent nodes
   2.2 Process the ∼neighbors
   2.3 Propagate only

Linearity achieved because:

- no global synchronisation
- canonic-based representation $=_c$
- PW's smart visit

First-order Check

Data: an initial state
Result: $\mathcal{F}ail$ or a final state

```
Procedure Main()
  foreach node n do
    if canonic(n) undefined
     then
      BuildClass(n)
    end
  end
```

```
Procedure Enqueue(m, c)
  case m, c of
    Abs(m′), Abs(c′) ⇒
      create edge m′ ∼ c′
    App(m1, m2), App(c1, c2) ⇒
      create edges m1 ∼ c1
       and m2 ∼ c2
    bVar(_), bVar(_) ⇒ ()
    fVar_, fVar(_) ⇒ ()
    _, _ ⇒ fail
  end
  canonic(m) := c
  queue(c).push(m)
```

```
Procedure BuildClass(c)
  canonic(c) := c
  visiting(c) := true
  queue(c) := {c}
  while queue(c) is non-empty do
    n := queue(c).pop()
    foreach parent m of n do
      case canonic(m) of
        undefined ⇒ BuildClass(m)
        c′ ⇒ if visiting(c′) then fail
      end
    end
    foreach ∼neighbour m of n do
      case canonic(m) of
        undefined ⇒ Enqueue(m, c)
        c′ ⇒ if c′ ≠ c then fail
      end
    end
  end
  visiting(c) := false
```

Variables Check

Data: canonic($\cdot$) representation of $\mathcal{Q}\#$
Result: is $\mathcal{Q}\#$ a sharing equivalence?

Procedure `VarsCheck()`

```
foreach variable node n do
    case n, canonic(n) of
        fVar(l), fVar(l') ⟹
            assert canonic(l) = canonic(l')
        bVar(x), bVar(y) ⟹
            assert x = y
    end
end
```

Conclusions

- Consequence: ⚡ β-conversion is bilinear ⚡
- A theory of sharing equality independent of algorithms
- A first / higher-order decomposition of the problem
- A linear PW-like algorithm for sharing equality
- We implemented the algorithm and verified its complexity
- On ArXiV: detailed proofs of correctness, completeness, and linearity

Thanks for your attention!