Choco - November 2019

Higher-Order Distributions for Linear Logic

Marie Kerjean (Inria Rennes -LS2N)
Based joints works with Jean-Simon Lemay (Oxford)

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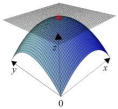
~~Higher-Order Distributions for Linear Logic~~
Differentiation and Duality (in denotational semantics)

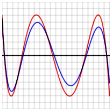
Marie Kerjean (Inria Rennes -LS2N)
Based joints works with Jean-Simon Lemay (Oxford), Christine Tasson (IRIF), Yoann Dabrowski (Lyon 1)

Curry-HOward for COmputing differentials

- As pure mathematicians study differentiation as a local and linear approximation of functions.

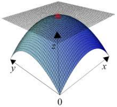


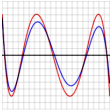
- As applied mathematicians we study and approximate infinite objects in numerical analysis.



- As logicians, what do we have to say about the computation of differentials ?

Curry-Howard-Lambek for Computing differentials





As logicians, what do we say to ~~death~~
the computation of differentials ?

The syntax mirrors the semantics.

Programs	**Logic**	**Semantics**
`fun (x:A)-> (t:B)`	Proof of $A \vdash B$	$f : A \to B$.
Types	Formulas	Objects
Execution	Cut-elimination	Equality
–	**DiLL**	**Functional Analysis**

The logic is :

- (linear) Classical : $A^\perp := A \multimap \perp$ and $A^{\perp\perp} \simeq A$.
- Higher-Order : $\lambda f.\lambda g.f(g)$.

The models should be:

- Reflexive. $A' := \mathcal{L}(A, \mathbb{R})$ and $A \simeq A''$
- Higher-Order. ~~$f : \mathbb{R}^n \to \mathbb{R}$~~ but $f : \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$

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I will present these result not necessarily in chronological order.

- Part I: Classical Smooth models of Differential Linear Logic.
- Part II: Higher-Order Smooth models of Differential Linear Logic.

Linear logic, once and for all

A linear implication

$$A \Rightarrow B = \, !A \multimap B$$
$$\mathcal{C}^\infty(A, B) \simeq \mathcal{L}(!A, B)$$

A focus on linearity

- Higher-Order is about *Seely's isomoprhism.*

$$\mathcal{C}^\infty(A \times B, C) \simeq \mathcal{C}^\infty(A, \mathcal{C}^\infty(B, C))$$
$$\mathcal{L}(!(A \times B), C) \simeq \mathcal{L}(!A, \mathcal{L}(!B, C))$$
$$!(A \times B) \simeq \, !A \hat{\otimes} !B$$

- Classicality is about a linear involutive negation :

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Just a glimpse at Differential Linear Logic

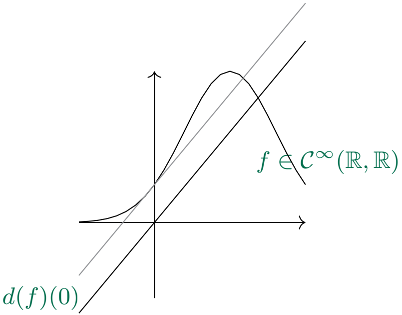
Differential Linear Logic

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}\, d$$

A linear proof is in particular non-linear.

$$\frac{f : !A \vdash B}{D_0(f) : A \vdash B}\, \bar{d}$$

From a non-linear proof we can extract a linear proof



Just a glimpse at Differential Linear Logic

$$A, B := A \otimes B | 1 | A \parr B | \bot | A \oplus B | 0 | A \times B | \top | !A | !A$$

Exponential rules of DiLL_0

$$\frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\, c \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\, w \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\, d$$

$$\frac{\vdash \Gamma, !A, \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\, \bar{c} \qquad \frac{\vdash}{\vdash !A}\, \bar{w} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\, \bar{d}$$

$\rightsquigarrow$ *A particular point of view on differentiation induced by duality.*

Normal functors, power series and λ-calculus. Girard, APAL(1988)

Differential interaction nets, Ehrhard and Regnier, TCS (2006)

Ok, just a little bit more

$$A, B := A \otimes B | 1 | A \parr B | \bot | A \oplus B | 0 | A \times B | \top | !A | !A$$

$$\underset{\textit{functions}}{[\![?A]\!] := \mathcal{C}^\infty([\![A]\!]', \mathbb{R})'} \qquad \underset{\textit{distributions}}{[\![!A]\!] := \mathcal{C}^\infty([\![A]\!], \mathbb{R})'}$$

Exponential rules of DiLL_0

$$\frac{\vdash \Gamma, f : ?A, g : ?A}{\vdash \Gamma, f.g : ?A}\, c \qquad \frac{\vdash \Gamma}{\vdash \Gamma, cst_0 : ?A}\, w \qquad \frac{\vdash \Gamma, \ell : A}{\vdash \Gamma, \ell : ?A}\, d$$

$$\frac{\vdash \Gamma, \phi : !A, \qquad \vdash \Delta, \psi : !A}{\vdash \Gamma, \Delta, \phi * \psi : !A}\, \bar{c} \qquad \frac{}{\vdash !A}\, \bar{w} \qquad \frac{\vdash \Gamma, v : A}{\vdash \Gamma, (f \mapsto D_0(f)) : !A}\, \bar{d}$$

Classical Models of Differential Linear Logic
in Functional Analysis.

A bit of context about linear logic and duality

Smoothness and Duality

Objectives

Spaces : E is a **locally convex** and **Haussdorf** topological vector space.
Functions: $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is infinitely and everywhere differentiable.

The two requirements works as opposite forces .

- ✓ A cartesian closed category with smooth functions.

  $\rightsquigarrow$ Completeness, and a dual topology fine enough.

- ✓ Interpreting $(E^\perp)^\perp \simeq E$ without an orthogonality:

  $\rightsquigarrow$ Reflexivity : $E \simeq E''$, and a dual topology coarse enough.

.

What's not working

A space of (non necessarily linear) functions between finite dimensional spaces is not finite dimensional.

$$\dim \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^m) = \infty.$$

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The tentative to have a normed space of analytic functions fails (Girard's Coherent Banach spaces).

- We want to use power series.
- For polarity reasons, we want the supremum norm on spaces of power series.
- But a power series can't be bounded on an unbounded space (Liouville's Theorem).
- Thus functions must depart from an open ball, but arrive in a closed ball. Thus they do not compose.
- This is why Coherent Banach spaces don't work.

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We can't restrict ourselves to normed spaces.

MLL in TopVect

It's a mess.

Duality is not an orthogonality in general :

- It depends of the topology E'_β, E'_c, E'_w, E'_μ on the dual.
- It is typically *not* preserved by $\otimes$.
- It is in the canonical case not an orthogonality : E'_β is not reflexive.

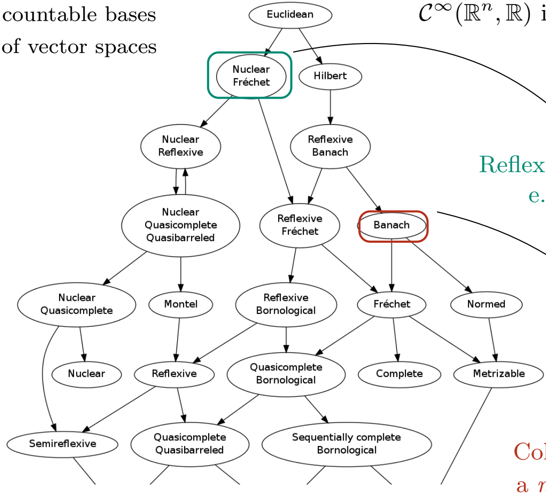
Monoidal closedness does not extends easily to the topological case :

- Many possible topologies on $\otimes$: $\otimes_\beta$, $\otimes_\pi$, $\otimes_\varepsilon$.
- $\mathcal{L}_\mathcal{B}(E \otimes_\mathcal{B} F, G) \simeq \mathcal{L}_\mathcal{B}(E, \mathcal{L}_\mathcal{B}(F, G))$
  $\Leftrightarrow$ "Grothendieck problème des topologies".

Which interpretation for formulas $\mathcal{L}$?

[Ehr02] [Ehr05] [DE08]
countable bases
of vector spaces

$\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is not finite dimensional



Reflexive anc complete :
e.g. $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$

Coherent Banach spaces [Gir99]
a *norm* is too restrictive

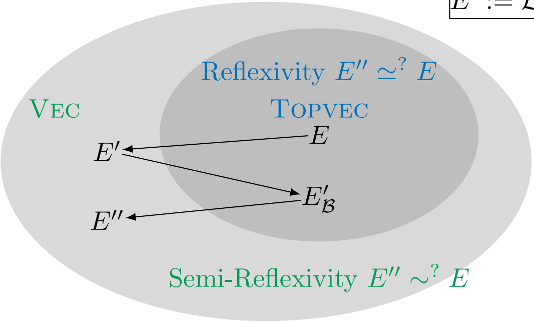
Smoothness and Duality

Smoothness

Spaces : $[\![A]\!]$ is a **locally convex** and **Haussdorf** topological vector space.
Functions: $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is infinitely and everywhere differentiable.

A coinductive definition : f is smooth iff it is differentiable and its differentials everywhere are smooth.

$E' := \mathcal{L}(E, \mathbb{R})$



In general, reflexive spaces enjoy `poor stability properties`.

Smoothness and Duality

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In general, reflexive spaces enjoy `poor stability properties`.

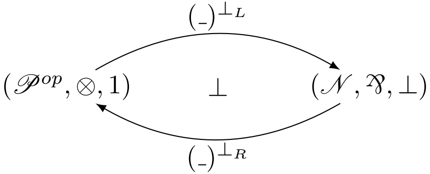
- No closure by $E \mapsto E''$.
- No stability by linear connectives $\otimes$, $⅋$, $- \multimap -$.

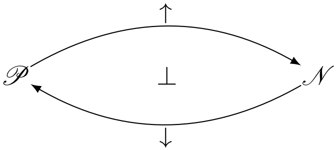
KEEP CALM AND POLARIZE

Chiralities: a categorical model for polarized MLL

Syntax

Negative Formulas: $N, M := a \mid ?P \mid N \parr M \mid \bot \mid N \,\&\, M \mid \top \mid$
Positive Formulas: $P, Q := a^{\bot} \mid !N \mid P \otimes Q \mid 0 \mid P \oplus Q \mid 1$



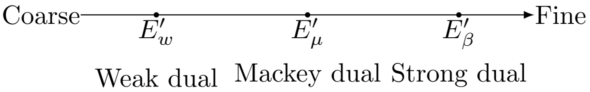


$$N^{\bot_R \bot_L} \simeq N$$
$$\mathcal{N}(\uparrow p, m \parr n) \simeq \mathcal{N}(\uparrow(p \otimes m^{\bot}), n)$$

Shopping for a good dual

The topology on your dual depends on the sets your functions are supposed to be uniformly convergent on :

$$f_n \to f \Leftrightarrow \forall \epsilon, \forall B, \exists N, n \geq N \Rightarrow |(f_n - f)(B)| < \epsilon.$$



- Weak reflexivity and Mackey reflexivity is immediate.
- Strong reflexivity is the traditional one and is much harder to attain. It decompose as:
  - the algebraic equality between E and $(E'_\beta)'$, equivalent to some weak completeness condition.
  - the topological correspondence $E \hookrightarrow (E'_\beta)'_\beta$, called barrelledness.

With the Weak Dual, a negative interpretation

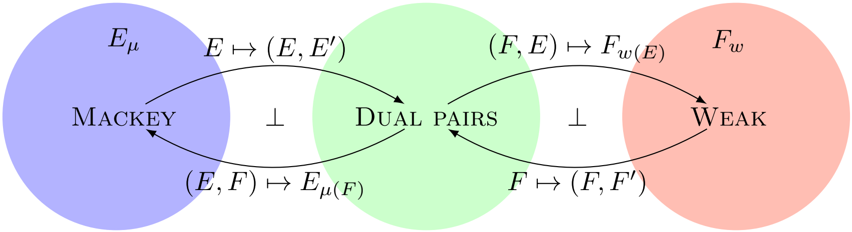
$$(\textsc{TopVec}, \otimes_w, \mathbb{R}) \quad \underset{(-)'_w}{\overset{(-)'_w}{\rightleftarrows}} \perp \quad (\textsc{Weak}^{op}, \invamp_w, \mathbb{R}) \qquad\qquad \textsc{TopVec} \underset{\iota}{\overset{(-)_w}{\rightleftarrows}} \perp \; \textsc{Weak}$$

in which ι denotes the inclusion functor.

Stability properties, "monoidal closedness".

K. *Weak topology for Linear Logic* LMCS. (2016)

The Mackey-Arens Theorem, by Barr



$$\mathcal{L}(E_\mu, F) = \mathcal{L}(E, F_w)$$

On ∗-autonomous categories of topological vector spaces, M. Barr Cahiers Topologie Géom. Différentielle Catég., 2000.

On convex topological vector spaces, G. Mackey, Trans. Amer. Math. Soc., 1946.

With the Mackey Dual, almost a positive interpretation

$$(\textsc{Mackey}, \otimes_\mu, \mathbb{R}) \ \underset{(-)'_\mu}{\overset{(-)'_\mu}{\rightleftarrows}}\!\!\!\!\!\!\!\!\perp\ \ \ (\textsc{TopVec}, \parr_\mu, \mathbb{R}) \qquad\qquad \textsc{Mackey} \ \underset{(-)_\mu}{\overset{\iota}{\rightleftarrows}}\!\!\!\!\perp\ \ \textsc{Weak}$$

in which ι denotes the inclusion functor.

Stability properties, but no "monoidal closedness".

With bornological spaces, a positive interpretation

Working with bounded sets instead of open sets : if E is bornological, then $\ell : E \to F$ is continuous if and only if $\ell(B)$ is bounded for every set B.

$$(\textsc{bTopVec}, \otimes_\beta, \mathbb{R}) \underset{(\mathsf{Born}((-)'_\mu))}{\overset{\mathsf{Top}(-)'_\mu}{\rightleftarrows}} \perp \; (\textsc{Mco}^{op}, \parr_b, \mathbb{R}) \qquad \textsc{bTopVec} \underset{\mathsf{Born}}{\overset{\mathsf{Top}}{\rightleftarrows}} \perp \; \textsc{TopVec}$$

Convenient differential category Blute, Ehrhard Tasson Cah. Geom. Diff. (2010)

Again bornological spaces

$$(\textsc{ubTopVec}, \hat{\otimes}_\beta^M, \mathbb{R}) \underset{(-)'_\mu}{\overset{\mathscr{S}((-)'_\mu)}{\rightleftarrows}} (\mathbf{Compl}\mu\mathbf{Sch}^{op}, \epsilon, \mathbb{R}) \quad (\perp)$$

$$\textsc{TopVec} \underset{\iota}{\overset{SS(-)}{\rightleftarrows}} \mathbf{Sch} \quad (\perp)$$

Models of Linear Logic based on Schwartz ε product. Dabrowski, K. 2018.

With the strong dual, a dialogue chirality

$$E \simeq (E'_\beta)'_\beta \Leftrightarrow E \text{ barrelled and } E \text{ weakly quasi complete.}$$

$$(\textsc{Barr}, \otimes_\beta, \mathbb{R}) \underset{(-)'_\mu}{\overset{(-)'_\sigma}{\rightleftarrows}} \perp \ (\textsc{wqCompl}^{op}, \parr_w, \mathbb{R}) \qquad \textsc{Mackey} \underset{(-)_\mu}{\overset{(-)_w}{\rightleftarrows}} \perp \ \textsc{Weak}$$

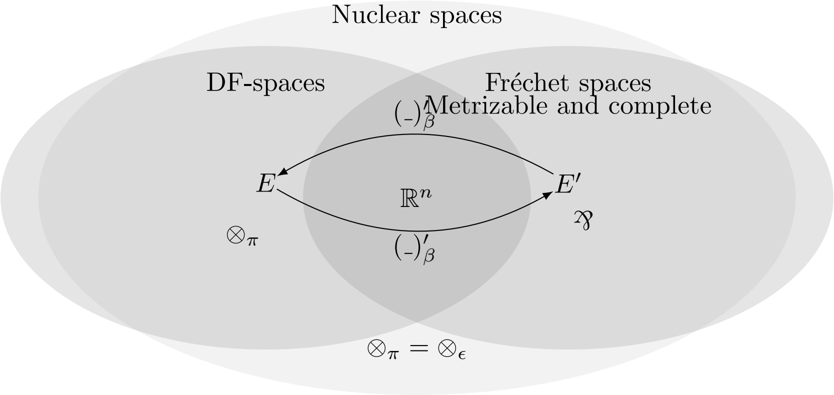
With Metric Spaces, a negative interpretation

$$(\textsc{Ndf}, \tilde{\otimes}_\pi, \mathbb{R}) \underset{(-)'_\beta}{\overset{(-)'_\beta}{\rightleftarrows}} (\textsc{Nf}^{op}, \hat{\otimes}, \mathbb{R}) \qquad \bot$$

$$\textsc{TopVec} \underset{\iota}{\overset{\tilde{-}}{\rightleftarrows}} \textsc{Compl} \qquad \bot$$

A logical account for LPDEs K. LICS2018

With Metric Spaces, a negative interpretation



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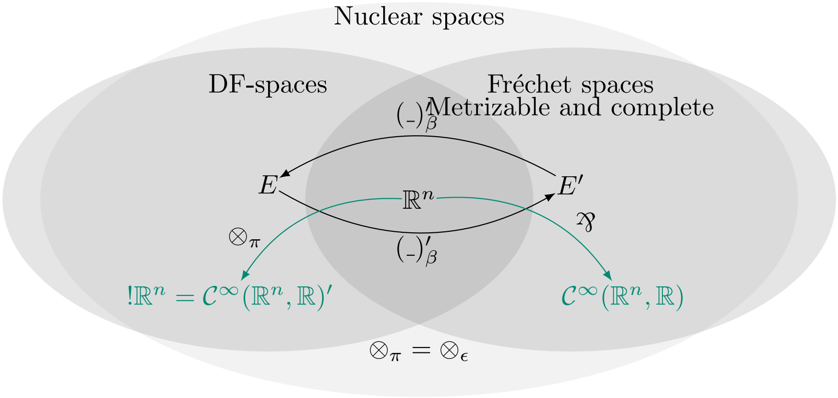


Higher-Order Smooth Models of Differential Linear Logic.

How to generalize distributions ?

Higher-Order is a story of approximation

"It soon becomes clear in thinking about "higher-types" [that] it also becomes necessary to introduce some idea of finite approximation "

Dana Scott, A Mathematical Theory of Computation.

What is surprising is that approximation allows cartesian closedness.

Approximation on negatives: power series.

$$(\textsc{TopVec}, \otimes_w, \mathbb{R}) \ \underset{(-)'_w}{\overset{(-)'_\sigma}{\rightleftarrows}}\!\!\perp\ (\textsc{Weak}^{op}, \parr_w, \mathbb{R}) \ \underset{U}{\overset{\mathcal{H}}{\rightleftarrows}}\!\!\perp\ (\textsc{Weak}^{\infty}, \times, \{0\})$$

where $\mathcal{H} : E \mapsto \prod_n \mathcal{H}^n(E, \mathbb{R})$, the space of formal power series, that is tuples of monomials. *Cartesian closedeness is inherited from combinatorial arguments and analytic functions.*

The idea of power series is pervasive in models of Differential Linear Logic:

- Köthe spaces [Ehrhard], a negative interpretation.
- Mackey spaces [Tasson, K.], an intuitionnistic interpretation focusing on negatives.
- Template Games [Mellies] ?

Approximation on positives: discretisation

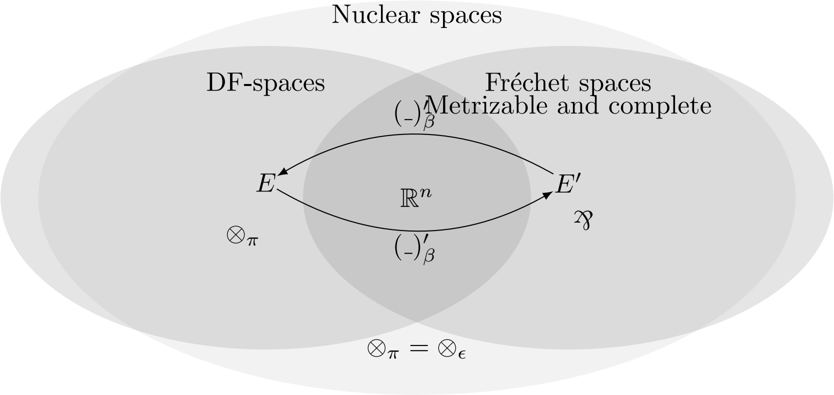
$$(\textsc{bTopVec}^{op}, \otimes_\beta, \mathbb{R}) \underset{(\mathsf{Born}((-)'_\mu))}{\overset{\mathsf{Top}(-)'_\mu}{\rightleftarrows}}\!\!\perp\; (\textsc{Mco}, \parr_b, \mathbb{R}) \underset{U}{\overset{\Delta}{\rightleftarrows}}\!\!\perp\; (\textsc{Mco}^\infty, \times, \{0\})$$

Where $\Delta : E \mapsto \overline{< \delta_x >_{x \in E}}$ considers that the only distributions that acts on smooth functions are the one which acts on a finite number of points.

Convenient differential category Blute, Ehrhard Tasson Cah. Geom. Diff. (2010)

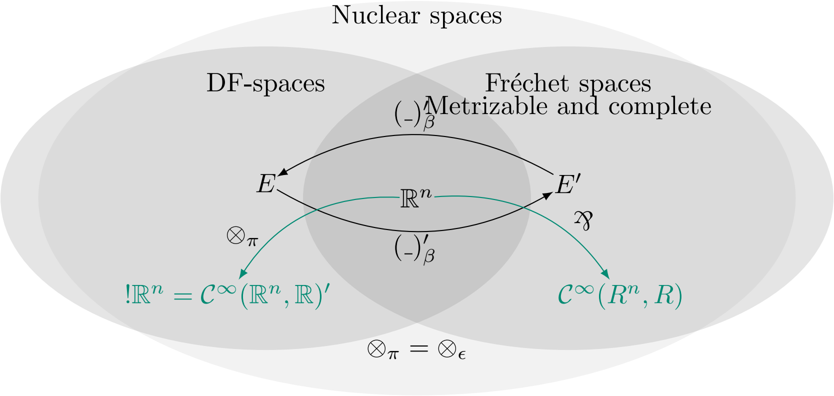
Higher-Order Distributions

With JS-Lemay (Oxford), we tackled higher-order models generalizing the nuclear Fréchet /DF Duality.



Higher-Order Distributions for DiLL Lemay & K. Fossacs 2019.

Higher-Order Distributions



$\mathcal{E}(\mathbb{R}^n) := \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ $\qquad$ $\mathcal{E}'(\mathbb{R}^n) := \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$

Distributions enjoy a Kernel theorem: $\mathcal{C}^\infty(E, \mathbb{R})' \hat{\otimes} \mathcal{C}^\infty(F, \mathbb{R})' \simeq \mathcal{C}^\infty(E \times F, \mathbb{R})'$.

Higher-Order Distributions for DiLL Lemay & K. Fossacs 2019.

Constructing some notion of smoothness which leaves stable the class of reflexive topological vector space.

We tackle this issue through the space of distribution

Consider E a topological vector space.

- Define an order on linear injections $f : \mathbb{R}^n \hookrightarrow E$ by $f \leq g := \exists \iota : \mathbb{R}^n \hookrightarrow \mathbb{R}^m, f = g \circ \iota$.
- Define the action of a distribution on E with respect to these linear injections:

$$\mathcal{E}'(E) := \varinjlim_{f:\mathbb{R}^n \multimap E} \mathcal{E}'_f(\mathbb{R}^n)$$

directed under the inclusion maps defined as

$$S_{f,g} : \mathcal{E}'_g(\mathbb{R}^n) \to \mathcal{E}'_f(\mathbb{R}^m), \phi \mapsto (h \mapsto \phi(h \circ \iota_{n,m}))$$

This is similar to work on $\mathcal{C}^\infty$-algebras [KainKrieglMichor87], which we need to refine to obtain reflexivity.

A good inductive limit

Because the distributions spaces with which we build the inductive limit are extremely regular, we have

- $\mathcal{E}'(E)$ is always reflexive.
- $\mathcal{E}'(E)$ is the dual of a projective limit of spaces of functions :

$$\mathcal{E}(E) := \varprojlim_{f:\mathbb{R}^n \multimap E} \mathcal{E}_f(\mathbb{R}^n)$$

$$\phi \in \mathcal{E}'(E) \text{ acts on } \mathbf{f} = (\mathbf{f}_f)_{f:\mathbb{R}^n \hookrightarrow E}.$$

where $\mathbf{f}_f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$.

The Kernel Theorem lifts to Higher-Order :

$$\mathcal{E}(E) \hat{\otimes} \mathcal{E}(F) \simeq \mathcal{E}(E \oplus F)$$

Reflexivity is enough for the structural morphisms

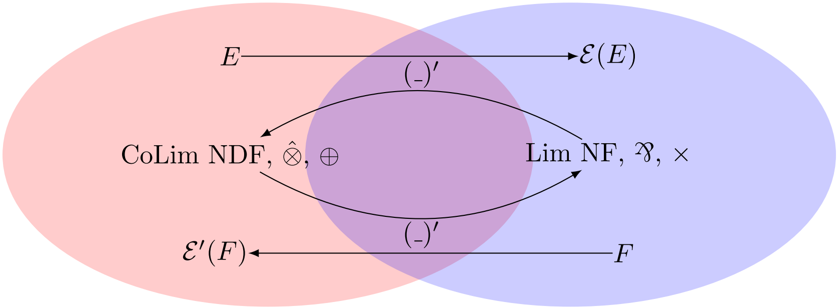
Because we worked with reflexive spaces at the beginning, we can built natural transformations :

$$d_E : \begin{cases} !(E) \to E'' \simeq E \\ \phi \mapsto (\underbrace{\ell}_{E \multimap \mathbb{R}} \in E' \mapsto \underbrace{\phi[(\overbrace{\ell \circ f}^{\mathbb{R}^n \to \mathbb{R}})_{f : \mathbb{R}^n \hookrightarrow E} \in \mathcal{E}(E)]}_{\mathbb{R}}) \end{cases}$$

$$\bar{d}_E : \begin{cases} E \to !E \simeq (\mathcal{E}(E))' \\ x \mapsto ((\mathbf{f}_f)_{f : \mathbb{R}^n \multimap E'}) \mapsto D_0 \mathbf{f}_f(f^{-1}(x)) \\ \quad \text{where } f \text{ is injective such that } x \in Im(f) \; . \end{cases}$$

And interpretations for (co)-weakening and (co)-contraction follow from the Kernel Theorem.

We have obtain polarized model of Differential Linear Logic :



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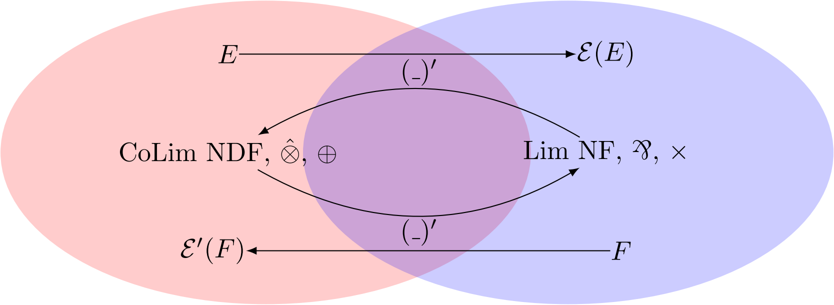

... without promotion $\dfrac{!\Gamma \vdash A}{!\Gamma \vdash !A}$

We didn't have a Cartesian Closed Category

This definition gives us functoriality only on isomorphisms :

$$! : \begin{cases} \textsc{Refl}_{iso} \to \textsc{Refl}_{iso} \\ E \mapsto \mathcal{E}'(E) \\ \ell : E \multimap F \mapsto !\ell \in \mathcal{E}(F') \end{cases}$$

where

$$(!\ell)(\phi)(\mathbf{g}) = \phi((\mathbf{g}_{\underbrace{\ell \circ f}_{\mathbb{R}^n \hookrightarrow F}})_{f:\mathbb{R}^n \hookrightarrow E}).$$

No category with smooth functions as maps.

We have however a good candidate to make a co-monad of our functor.

$$\mu_E : \begin{cases} !E \to !!E \\ \phi \mapsto \left((\mathbf{g}_g)_g \in \mathcal{E}(!E) \simeq \varinjlim_g \mathcal{C}^\infty_g(\mathbb{R}^m) \right) \mapsto \mathbf{g}_g(g^{-1}(\phi)) \\ \quad \text{when } \phi \in Im(g) \text{ and } g \text{ is injective} \end{cases}$$

Thanks Tom Hirschowitz for the remark !

Functoriality, but no associativity

Functoriality is obtained through an epi-mono decomposition. Consider $\ell \in \mathcal{L}(E, F)$:

$$\ell = E \xrightarrow{\pi_\ell} E/Ker(\ell) \xrightarrow{\hat{\ell}} F.$$

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This gives us functoriality, naturality of d, $\bar{d}$ and μ but not assoiative composition between non-linear functions.

$\rightsquigarrow$ *Conclusion: a tentative abstract formulation to approximation techniques.*

Conclusion

Differential Linear Logic and its semantics shows the relevance of duality for differentiation

- Not a surprise for semantics/distributions.
- But interesting for programming ? [Brunel,Mazza,Pagani POPL'20] [Dialectica]

Perspectives:

- Can we adapt results of approximation theory to models of DiLL ?
- Will this be in any help for the generalisation of the linear/non-linear interaction to the one of the solution/parameter of differential equations ?
- Should the linear/non-linear interaction follow the pattern of the positive/negative one ?