MONADS, EQUATIONAL THEORIES AND METRICS FOR NONDETERMINISTIC AND PROBABILISTIC SYSTEMS

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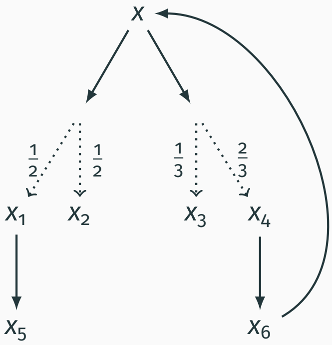
x
$\frac{1}{2}$
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x_1
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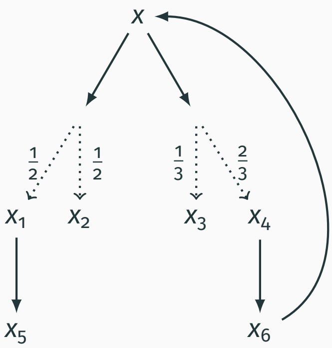
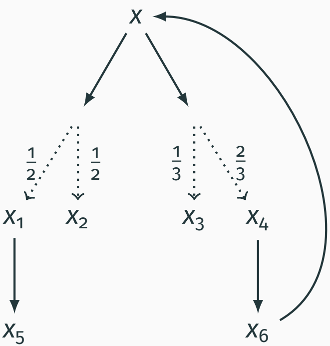
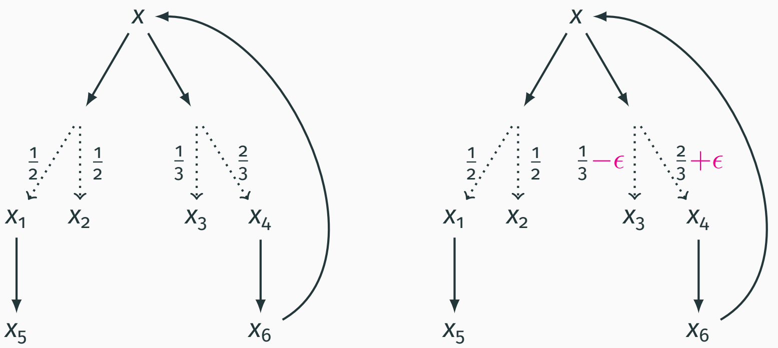


- nondeterminism and probability as computational effects: monads and equational theories



- nondeterminism and probability as computational effects: monads and equational theories
- equational reasoning in program semantics

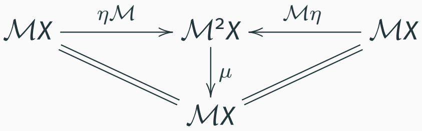


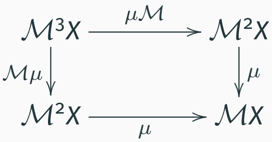
- nondeterminism and probability as computational effects: monads and equational theories
- equational reasoning in program semantics
- program distance ⇒ monads on metric spaces and quantitative equational theories

Monads and Equational Theories for Nondeterminism and Probability

Monad $(\mathcal{M}, \eta, \mu)$
in Sets

- functor $\mathcal{M} : X \mapsto \mathcal{M}(X)$
- unit $\eta_X : X \to \mathcal{M}(X)$
- multiplication $\mu_X : \mathcal{M}\mathcal{M}(X) \to \mathcal{M}(X)$





Monad $(\mathcal{M}, \eta, \mu)$
in Sets

Equational Theory (Σ, E)

- Σ a signature
- E a set of equations

- equations $t = s$
- deductive system: equational logic
  $$\{ t = s, s = u \} \vdash t = u$$
- models: algebras (A, Σ^A) satisfying the equations

Monad $(\mathcal{M}, \eta, \mu)$
in Sets

$\Longleftrightarrow$

Equational Theory (Σ, E)

- Σ a signature
- E a set of equations

(Σ, E) is a presentation of $(\mathcal{M}, \eta, \mu)$

The category **EM**$(\mathcal{M})$ of Eilenberg-Moore algebras for $(\mathcal{M}, \eta, \mu)$ is isomorphic to the category **A**(Σ, E) of algebras (models) of (Σ, E)

Category **EM**$(\mathcal{M})$

- objects: $(A, \alpha : \mathcal{M}(A) \to A)$ with α commuting with η, μ
- arrows: algebra morphisms

Category **A**(Σ, E)

- objects: models (A, Σ^A) of (Σ, E)
- arrows: homomorphisms of (Σ, E)-algebras

Monad $(\mathcal{M}, \eta, \mu)$
in Sets

$\Longleftrightarrow$

Equational Theory (Σ, E)

- Σ a signature
- E a set of equations

(Σ, E) **is a presentation of** $(\mathcal{M}, \eta, \mu)$

The category $\mathbf{EM}(\mathcal{M})$ of Eilenberg-Moore algebras for $(\mathcal{M}, \eta, \mu)$ is isomorphic to the category $\mathbf{A}(\Sigma, E)$ of algebras (models) of (Σ, E)

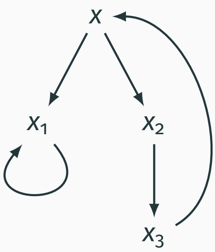
Corollary:

$$\mathcal{M}(X) \cong \mathit{Terms}(X, \Sigma)_{/E}$$

EXAMPLE: NONDETERMINISM

Monad $(\mathcal{M}, \eta, \mu)$ in Sets $\Longleftrightarrow$ Equational Theory (Σ, E)

- Σ a signature
- E a set of equations



$c : X \to \mathcal{P}(X)$

$c(x) = \{x_1, x_2\}$
$c(x_1) = \{x_1\}$
...

Monad $(\mathcal{M}, \eta, \mu)$
in Sets

$\iff$

Equational Theory (Σ, E)

- Σ a signature
- E a set of equations

Powerset (non-empty) monad $(\mathcal{P}, \eta, \mu)$

- $\mathcal{P} : X \mapsto \{S \mid S \text{ is a non-empty, finite subset of } X\}$
- $\eta : x \mapsto \{x\}$
- $\mu : \{S_1, \ldots, S_n\} \mapsto \bigcup_i S_i$

$\iff$

Equational theory of semilattices

- Σ = binary operation $\oplus$
- axioms of E =

$$
\begin{aligned}
(x \oplus y) \oplus z &\overset{(A)}{=} x \oplus (y \oplus z) \\
x \oplus y &\overset{(C)}{=} y \oplus x \\
x \oplus x &\overset{(I)}{=} x
\end{aligned}
$$

Monad $(\mathcal{M}, \eta, \mu)$
in Sets

$\Longleftrightarrow$

Equational Theory (Σ, E)

- Σ a signature
- E a set of equations

Powerset (non-empty)
monad $(\mathcal{P}, \eta, \mu)$

- $\mathcal{P} : X \mapsto \{S \mid S$ is a non-empty, finite subset of $X\}$
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\end{aligned}
$$

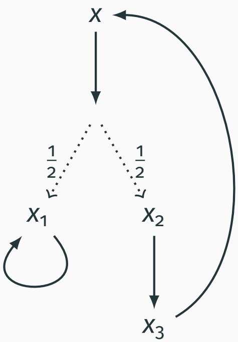
Corollary:
$\mathcal{P}(X) \cong Terms(X, \Sigma)/_E$

Monad $(\mathcal{M}, \eta, \mu)$
in Sets

$\iff$

Equational Theory (Σ, E)

- Σ a signature
- E a set of equations



$c : X \to \mathcal{D}(X)$

$c(x) = \frac{1}{2}\, x_1 + \frac{1}{2}\, x_2$
$c(x_1) = 1\, x_1$
...

Monad $(\mathcal{M}, \eta, \mu)$ in Sets $\Longleftrightarrow$ Equational Theory (Σ, E)

- Σ a signature
- E a set of equations

Distribution monad $(\mathcal{D}, \eta, \mu)$

- $\mathcal{D} : X \mapsto \{\Delta \mid \Delta$ is a finitely supported probability distribution on $X\}$
- $\eta : x \mapsto 1x$
- $\mu : \sum_i p_i \Delta_i \mapsto \sum_i p_i \cdot \Delta_i$

$\Longleftrightarrow$

Equational theory of convex algebras

- Σ = binary operations $+_p$ for all $p \in (0, 1)$
- axioms of E =

$$(x +_q y) +_p z \overset{(A_p)}{=} x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z)$$

$$x +_p y \overset{(C_p)}{=} y +_{1-p} x$$

$$x +_p x \overset{(I_p)}{=} x$$

EXAMPLE: PROBABILITY

Monad $(\mathcal{M}, \eta, \mu)$
in Sets

$\Longleftrightarrow$

Equational Theory (Σ, E)

- Σ a signature
- E a set of equations

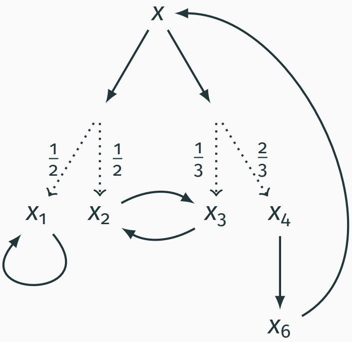
nondeterminism

+

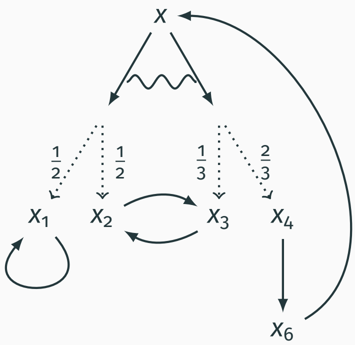
probability

=

?



- a transition reaches a set of probability distributions $\{ \frac{1}{2}x_1 + \frac{1}{2}x_2, \; \frac{1}{3}x_3 + \frac{2}{3}x_4 \}$
- Problem: $\mathcal{P} \circ \mathcal{D}$ is not a monad [Varacca, Winskel 2006]



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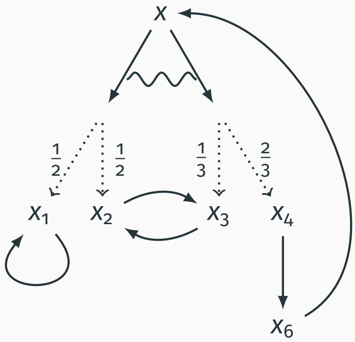
- Problem: $\mathcal{P} \circ \mathcal{D}$ is not a monad [Varacca, Winskel 2006]

Solution: use **convex sets of probability distributions**

$$\{ \tfrac{1}{2}x_1 + \tfrac{1}{2}x_2, \ldots, \tfrac{1}{4}x_1 + \tfrac{1}{4}x_2 + \tfrac{1}{6}x_3 + \tfrac{1}{3}x_4, \ldots, \tfrac{1}{3}x_3 + \tfrac{2}{3}x_4 \}$$



- a transition reaches a set of probability distributions $\{ \frac{1}{2}x_1 + \frac{1}{2}x_2,\ \frac{1}{3}x_3 + \frac{2}{3}x_4 \}$
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Solution: use **convex sets of probability distributions**

$$\{ \tfrac{1}{2}x_1 + \tfrac{1}{2}x_2, \ldots, \tfrac{1}{4}x_1 + \tfrac{1}{4}x_2 + \tfrac{1}{6}x_3 + \tfrac{1}{3}x_4, \ldots, \tfrac{1}{3}x_3 + \tfrac{2}{3}x_4 \}$$

+ accounts for probabilistic schedulers

The monad $(\mathcal{C}, \eta, \mu)$ in Sets:

- $\mathcal{C} : X \mapsto \{S \mid S$ is a non-empty, convex-closed, finitely generated set of finitely supported probability distributions over $X\}$
- $\eta_X : X \to \mathcal{C}(X)$

$$\eta_X : x \mapsto \{\ 1x\ \}$$

- $\mu_X : \mathcal{C}\mathcal{C}(X) \to \mathcal{C}(X)$

$$\mu_X : \bigcup_i \{\Delta_i\} \mapsto \bigcup_i \mathtt{WMS}(\Delta_i)$$

with $\mathtt{WMS} : \mathcal{D}\mathcal{C}(X) \to \mathcal{C}(X)$ the *weighted Minkowski sum*

$$\mathtt{WMS}(\sum_{i=1}^{n} p_i S_i) = \{\sum_{i=1}^{n} p_i \cdot \Delta_i \mid \text{for each } 1 \le i \le n, \Delta_i \in S_i\}$$

[Jacobs 2008 ...]

Monad $(\mathcal{M}, \eta, \mu)$
in Sets

$\Leftrightarrow$

Equational Theory (Σ, E)

- Σ a signature
- E a set of equations

Convex sets (non-empty)
of distributions
monad $(\mathcal{C}, \eta, \mu)$

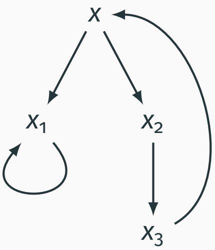
$\Leftrightarrow$

Equational theory of convex semilattices

- $\Sigma = \oplus$ and $+_p$ for all $p \in (0, 1)$
- axioms E :
  - axioms of semilattices
  - axioms of convex algebras
  - distributivity axiom (D)
    $(x \oplus y) +_p z \stackrel{(D)}{=} (x +_p z) \oplus (y +_p z)$

[Bonchi, Sokolova, V. 2019 and 2020]

Termination and Equational Reasoning in Program Semantics



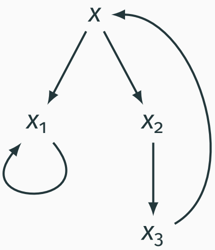
$c : X \to \mathcal{P}(X)$

$c(x) = \{x_1, x_2\}$
$c(x_1) = \{x_1\}$
...

Bisimulation equivalence (= coalgebraic behavioral equivalence)

$\exists$ R such that xRy implies:

- if $x \to x'$ then $y \to y'$ and $x'Ry'$
- symmetric



$c : X \to \mathcal{P}(X)$

$c(x) = \{x_1, x_2\}$
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Bisimulation equivalence (= coalgebraic behavioral equivalence)

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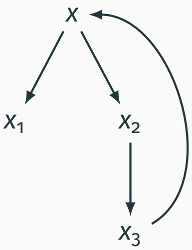
- symmetric

trivial if all states can move $\Rightarrow$ we need some observable: termination



$c : X \to \mathcal{P}^{\emptyset}(X)$

$c(x) = \{x_1, x_2\}$
$c(x_1) = \emptyset$ i.e., $x_1 \not\to$
...

Bisimulation equivalence (= coalgebraic behavioral equivalence)

$\exists$ R such that xRy implies:

- if $x \to x'$ then $y \to y'$ and $x'Ry'$
- symmetric

trivial if all states can move $\Rightarrow$ we need some observable: termination

Monad $(\mathcal{M}, \eta, \mu)$
in Sets

$\Longleftrightarrow$

Equational Theory (Σ, E)

- Σ a signature
- E a set of equations

Powerset (possibly empty) monad $(\mathcal{P}^{\emptyset}, \eta, \mu)$

- $\mathcal{P}^{\emptyset} : X \mapsto \{S \mid S \text{ is a finite subset of } X\}$
- $\eta : x \mapsto \{x\}$
- $\mu : \{S_1, \ldots, S_n\} \mapsto \bigcup_i S_i$

$\Longleftrightarrow$

Equational theory of semilattices with bottom

- $\Sigma = \star, \oplus$
- axioms of E=
  - axioms of semilattices

$$(x \oplus y) \oplus z \overset{(A)}{=} x \oplus (y \oplus z)$$
$$x \oplus y \overset{(C)}{=} y \oplus x$$
$$x \oplus x \overset{(I)}{=} x$$

  - bottom axiom $x \oplus \star = x$

Monad $(\mathcal{M}, \eta, \mu)$
in Sets

$\Leftrightarrow$

Equational Theory (Σ, E)

- Σ a signature
- E a set of equations

Convex sets
(possibly empty)
of distributions
monad $(\mathcal{C}^{\emptyset}, \eta, \mu)$

$\Leftrightarrow$

Equational theory of convex semilattices
with bottom and black-hole

- $\Sigma = \star, \oplus, +_p$ for all $p \in (0, 1)$
- axioms of $E =$
  - axioms of convex semilattices
  - bottom axiom $x \oplus \star = x$
  - black-hole axiom $x +_p \star = \star$

[Mio, Sarkis, V. 2021]

Processes $P ::= \textbf{nil} \mid a.P \mid P_1 \mathbin{\overline{\oplus}} P_2 \mid P_1 \mathbin{\overline{+}_p} P_2$

Semantics $c : Proc \to Terms(Proc, \Sigma)$

with $\Sigma = \star, \oplus, +_p$

$$\textbf{nil}\ \frac{}{\textbf{nil} \to \star} \qquad \overline{\oplus}\ \frac{P_1 \to t_1 \qquad P_2 \to t_2}{P_1 \mathbin{\overline{\oplus}} P_2 \to t_1 \oplus t_2}$$

$$\text{action}\ \frac{}{a.P \to P} \qquad \overline{+}_p\ \frac{P_1 \to t_1 \qquad P_2 \to t_2}{P_1 \mathbin{\overline{+}_p} P_2 \to t_1 +_p t_2}$$

APPLICATION: EQUATIONAL REASONING ON PROCESSES

Processes $P ::= \mathbf{nil} \mid a.P \mid P_1 \overline{\oplus} P_2 \mid P_1 \overline{+}_p P_2$

Semantics $c : \mathit{Proc} \to \mathit{Terms}(\mathit{Proc}, \Sigma)$

with $\Sigma = \star, \oplus, +_p$

$$(a.(\mathbf{nil} \overline{\oplus} \mathbf{nil}) \overline{+}_{\frac{1}{3}} a^3.\mathbf{nil}) \overline{\oplus} (a^2.\mathbf{nil} \overline{+}_{\frac{1}{2}} a.\mathbf{nil})$$

$$\downarrow$$

$$((\mathbf{nil} \overline{\oplus} \mathbf{nil}) +_{\frac{1}{3}} a^2.\mathbf{nil}) \oplus (a.\mathbf{nil} +_{\frac{1}{2}} \mathbf{nil})$$

Processes $P ::= \mathbf{nil} \mid a.P \mid P_1 \overline{\oplus} P_2 \mid P_1 \overline{+}_p P_2$

Semantics $c : \mathit{Proc} \to \mathit{Terms}(\mathit{Proc}, \Sigma)_{/E}$

with $\Sigma = \star, \oplus, +_p$

$$(a.(\mathbf{nil} \overline{\oplus} \mathbf{nil}) \overline{+}_{\frac{1}{3}} a^3.\mathbf{nil}) \overline{\oplus} (a^2.\mathbf{nil} \overline{+}_{\frac{1}{2}} a.\mathbf{nil})$$

$$\downarrow$$

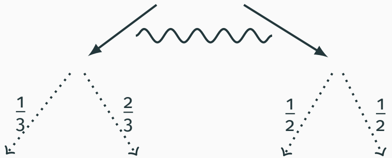
$$[((\mathbf{nil} \overline{\oplus} \mathbf{nil}) +_{\frac{1}{3}} a^2.\mathbf{nil}) \oplus (a.\mathbf{nil} +_{\frac{1}{2}} \mathbf{nil})]_{/E}$$

Processes $\quad P ::= \mathbf{nil} \mid a.P \mid P_1 \mathbin{\overline{\oplus}} P_2 \mid P_1 \mathbin{\overline{+}_p} P_2$

Semantics $\quad c : \mathit{Proc} \to \mathit{Terms}(\mathit{Proc}, \Sigma)_{/E} \simeq \mathcal{C}^{\emptyset}(\mathit{Proc})$

$$\text{with } \Sigma = \star, \oplus, +_p$$

$$(a.(\mathbf{nil} \mathbin{\overline{\oplus}} \mathbf{nil}) \mathbin{\overline{+}_{\frac{1}{3}}} a^3.\mathbf{nil}) \mathbin{\overline{\oplus}} (a^2.\mathbf{nil} \mathbin{\overline{+}_{\frac{1}{2}}} a.\mathbf{nil})$$



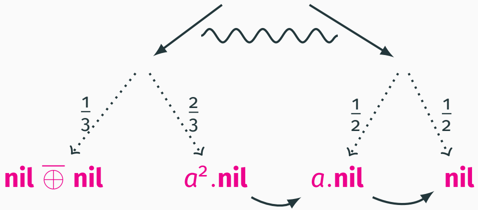
$\mathbf{nil} \mathbin{\overline{\oplus}} \mathbf{nil} \qquad a^2.\mathbf{nil} \qquad a.\mathbf{nil} \qquad \mathbf{nil}$

Processes $\quad P ::= \mathbf{nil} \mid a.P \mid P_1 \mathbin{\overline{\oplus}} P_2 \mid P_1 \mathbin{\overline{+}_p} P_2$

Semantics $\quad c : \mathit{Proc} \to \mathit{Terms}(\mathit{Proc}, \Sigma)_{/E} \simeq \mathcal{C}^{\emptyset}(\mathit{Proc})$

$$\text{with } \Sigma = \star, \oplus, +_p$$

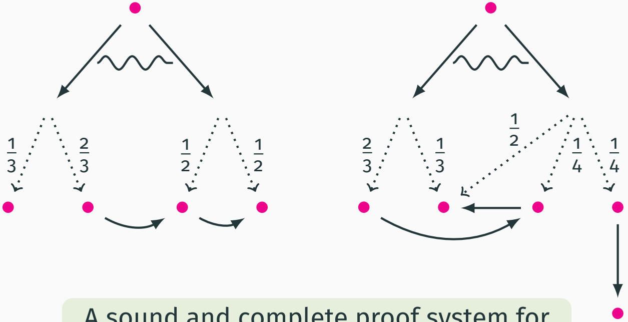
$$(a.(\mathbf{nil} \mathbin{\overline{\oplus}} \mathbf{nil}) \mathbin{\overline{+}_{\frac{1}{3}}} a^3.\mathbf{nil}) \mathbin{\overline{\oplus}} (a^2.\mathbf{nil} \mathbin{\overline{+}_{\frac{1}{2}}} a.\mathbf{nil})$$



Processes $P ::= \mathbf{nil} \mid a.P \mid P_1 \overline{\oplus} P_2 \mid P_1 \overline{+}_p P_2$

Semantics $c : \mathit{Proc} \to \mathit{Terms}(\mathit{Proc}, \Sigma)_{/E} \simeq \mathcal{C}^{\emptyset}(\mathit{Proc})$

with $\Sigma = \star, \oplus, +_p$



A sound and complete proof system for bisimulation equivalence on processes

$P \sim P' \quad \text{iff} \quad P \overset{E}{\equiv} P'$

AXIOMS FOR TERMINATION: THE ISSUE WITH BLACK-HOLE

Processes $P ::= \mathbf{nil} \mid a.P \mid P_1 \overline{\oplus} P_2 \mid P_1 \overline{+}_p P_2$

Semantics $c : \mathit{Proc} \to \mathit{Terms}(\mathit{Proc}, \Sigma)_{/E} \simeq \mathcal{C}^{\emptyset}(\mathit{Proc})$

with $\Sigma = \star, \oplus, +_p$

Bottom axiom $x \oplus \star = x$

Black-hole axiom $x +_p \star = \star$

AXIOMS FOR TERMINATION: THE ISSUE WITH BLACK-HOLE

Processes $P ::= \mathbf{nil} \mid a.P \mid P_1 \overline{\oplus} P_2 \mid P_1 \overline{+}_p P_2$

Semantics $c : \mathit{Proc} \to \mathit{Terms}(\mathit{Proc}, \Sigma)_{/E} \simeq \mathcal{C}^{\emptyset}(\mathit{Proc})$

with $\Sigma = \star, \oplus, +_p$

Bottom axiom $x \oplus \star = x$

Black-hole axiom $x +_p \star = \star$

$a.\mathbf{nil} \; \overline{+}_{\frac{1}{2}} \; \mathbf{nil} \longrightarrow [\mathbf{nil} +_{\frac{1}{2}} \star]_{/E}$

$\mathbf{nil} \longrightarrow [\star]_{/E}$

AXIOMS FOR TERMINATION: THE ISSUE WITH BLACK-HOLE

Processes $P ::= \mathbf{nil} \mid a.P \mid P_1 \overline{\oplus} P_2 \mid P_1 \overline{+}_p P_2$

Semantics $c : \mathit{Proc} \to \mathit{Terms}(\mathit{Proc}, \Sigma)_{/E} \simeq \mathcal{C}^{\emptyset}(\mathit{Proc})$

with $\Sigma = \star, \oplus, +_p$

Bottom axiom $x \oplus \star = x$

Black-hole axiom $x +_p \star = \star$

$$a.\mathbf{nil} \overline{+}_{\frac{1}{2}} \mathbf{nil} \quad \sim \quad \mathbf{nil}$$

$$\downarrow \qquad\qquad \downarrow$$

$$[\mathbf{nil} +_{\frac{1}{2}} \star]_{/E} \quad = \quad [\star]_{/E}$$

Monad $(\mathcal{M}, \eta, \mu)$
in Sets

$\Leftrightarrow$

Equational Theory (Σ, E)

- Σ a signature
- E a set of equations

$\perp$-closed convex sets
(possibly empty)
of subdistributions
monad $(\mathcal{C}^{\downarrow}, \eta, \mu)$

$\Leftrightarrow$

Equational theory of convex semilattices
with bottom

- $\Sigma = \star, \oplus, +_p$ for all $p \in (0, 1)$
- axioms E :
  - axioms of convex semilattices
  - bottom axiom $\quad x \oplus \star = x$

[Mio, Sarkis, V. 2021]

- subdistribution = $\sum_i p_i x_i$ with $\sum_i p_i \leq 1$
- S is $\perp$-closed = if $\sum_i p_i x_i \in S$ then $\sum_i q_i x_i \in S$ with $q_i \leq p_i$

BOTTOM ONLY SEMANTICS

Processes $P ::= \mathbf{nil} \mid a.P \mid P_1 \overline{\oplus} P_2 \mid P_1 \overline{+}_p P_2$

Semantics $c : \mathit{Proc} \to \mathit{Terms}(\mathit{Proc}, \Sigma)_{/E} \simeq \mathcal{C}^{\downarrow}(\mathit{Proc})$

with $\Sigma = \star, \oplus, +_p$

Bottom axiom $x \oplus \star = x$

Processes $P ::= \mathbf{nil} \mid a.P \mid P_1 \overline{\oplus} P_2 \mid P_1 \overline{+}_p P_2$

Semantics $c : Proc \to Terms(Proc, \Sigma)_{/E} \simeq \mathcal{C}^{\downarrow}(Proc)$

with $\Sigma = \star, \oplus, +_p$

Bottom axiom $x \oplus \star = x$

$$a.\mathbf{nil} \overline{+}_{\frac{1}{2}} \mathbf{nil} \not\simeq \mathbf{nil}$$

$$[\mathbf{nil} +_{\frac{1}{2}} \star]_{/E} \neq [\star]_{/E}$$

Processes $\quad P ::= \mathbf{nil} \mid a.P \mid P_1 \mathbin{\overline{\oplus}} P_2 \mid P_1 \mathbin{\overline{+}_p} P_2$

Semantics $\quad c : \mathit{Proc} \to \mathit{Terms}(\mathit{Proc}, \Sigma)_{/E} \;\simeq\; \mathcal{C}^{\downarrow}(\mathit{Proc})$

with $\Sigma = \star, \oplus, +_p$

Bottom axiom $\quad x \oplus \star = x$

$$a.\mathbf{nil} \mathbin{\overline{+}_{\frac{1}{2}}} \mathbf{nil} \quad \not\simeq \quad \mathbf{nil}$$

$$a.\mathbf{nil} \mathbin{\overline{+}_{\frac{1}{2}}} \mathbf{nil} \longrightarrow \;\overset{\frac{1}{2}\downarrow}{\dashrightarrow}\; \mathbf{nil}$$

RECAP

Convex sets (non-empty) of distributions monad $(\mathcal{C}, \eta, \mu)$ $\Leftrightarrow$ Equational theory of convex semilattices

Convex sets (possibly empty) of distributions monad $(\mathcal{C}^{\emptyset}, \eta, \mu)$ $\Leftrightarrow$ Equational theory of convex semilattices with bottom $x \oplus \star = x$ and black-hole $x +_p \star = \star$

$\bot$-closed convex sets (possibly empty) of subdistributions monad $(\mathcal{C}^{\downarrow}, \eta, \mu)$ $\Leftrightarrow$ Equational theory of convex semilattices with bottom $x \oplus \star = x$

Monads on Metric Spaces and Program Distances

Category Met of Metric Spaces:

- objects (X, d)
- morphisms $f : (X, d) \to (Y, d')$ are non-expansive maps between metric spaces

$$f : X \to Y \quad \text{with} \quad d'(f(x_1), f(x_2)) \leq d(x_1, x_2)$$

Monad $(\mathcal{M}, \eta, \mu)$ in Sets lifted to a monad $(\hat{\mathcal{M}}, \hat{\eta}, \hat{\mu})$ in Met, with:

- $\hat{\mathcal{M}} : (X, d) \mapsto (\mathcal{M}(X), \mathrm{lift}_{\mathcal{M}}(d))$
- $\hat{\eta}_{(X,d)} : (X, d) \to (\mathcal{M}(X), \mathrm{lift}_{\mathcal{M}}(d))$ and $\hat{\mu}_{(X,d)} : (\mathcal{M}\mathcal{M}(X),\ \mathrm{lift}_{\mathcal{M}}\mathrm{lift}_{\mathcal{M}}(d)) \to (\mathcal{M}(X), \mathrm{lift}_{\mathcal{M}}(d))$ non-expansive

The monad $(\mathcal{C}, \eta, \mu)$ of convex sets of distributions can be lifted to a monad $(\hat{\mathcal{C}}, \hat{\eta}, \hat{\mu})$ in Met:

- $\hat{\mathcal{C}} : (X, d) \mapsto (\mathcal{C}(X), HK(d))$

$HK(d) =$ Hausdorff-Kantorovich lifting of d

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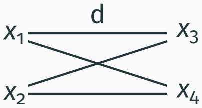

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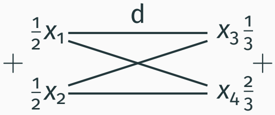

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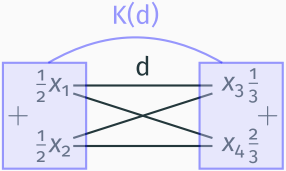

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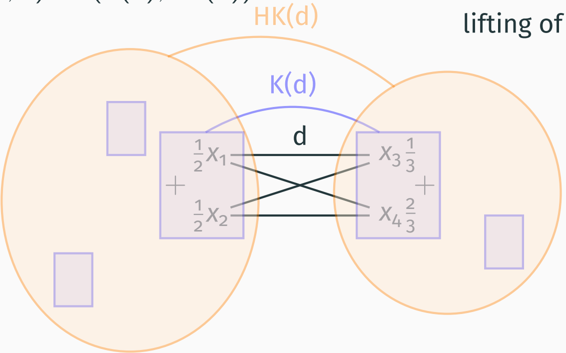

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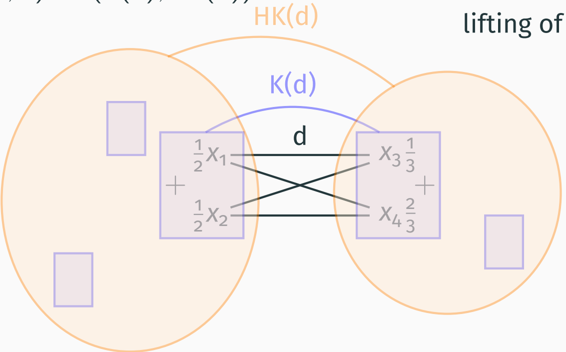

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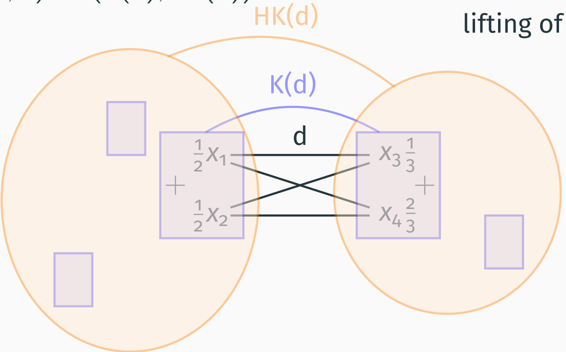

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  non-expansive

[Mio, V. 2020]

Monad $(\hat{\mathcal{M}}, \hat{\eta}, \hat{\mu})$ in Met

$\Leftrightarrow$

Quantitative Equational Theory (Σ, Q)

- Σ a signature
- Q a set of quantitative inferences

Monad $(\hat{\mathcal{M}}, \hat{\eta}, \hat{\mu})$ in Met $\Leftrightarrow$ Quantitative Equational Theory (Σ, Q)

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- Q a set of quantitative inferences

- quantitative equations $\quad t =_\epsilon t'$
- quantitative inferences

$$\{t_i =_{\epsilon_i} s_i\}_{i \in I} \vdash t =_\epsilon s$$

- quantitative deduction system

(Reflexivity) $\emptyset \vdash t =_0 t$

(Symmetry) $\{t =_\epsilon s\} \vdash s =_\epsilon t$

(Triangular) $\{t =_{\epsilon_1} u, u =_{\epsilon_2} s\} \vdash t =_{\epsilon_1 + \epsilon_2} s$

- models: quantitative algebras (A, Σ^A, d) satisfying the quantitative inferences

[Mardare, Panangaden, Plotkin 2016]

Monad $(\hat{\mathcal{M}}, \hat{\eta}, \hat{\mu})$
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Quantitative Equational Theory (Σ, Q)

- Σ a signature
- Q a set of quantitative inferences

(Σ, Q) **is a presentation of** $(\hat{\mathcal{M}}, \hat{\eta}, \hat{\mu})$

The category **EM**$(\hat{\mathcal{M}})$ of Eilenberg-Moore algebras for $(\hat{\mathcal{M}}, \hat{\eta}, \hat{\mu})$ is isomorphic to the category **A**(Σ, E) of quantitative algebras (models) of (Σ, Q)

THE QUANTITATIVE EQUATIONAL THEORY OF CONVEX SEMILATTICES

Monad $(\hat{\mathcal{M}}, \hat{\eta}, \hat{\mu})$ in Met $\Leftrightarrow$ Quantitative Equational Theory (Σ, Q)

- Σ a signature
- Q a set of quantitative inferences

Convex sets (non-empty) of distributions monad $(\hat{\mathcal{C}}, \hat{\eta}, \hat{\mu})$ in Met $\Leftrightarrow$ Quantitative equational theory of convex semilattices

- $\Sigma = \oplus$ and $+_p$ for all $p \in (0, 1)$
- quantitative inferences $Q =$

• axioms of convex semilattices,
with $t = t'$ becoming $\emptyset \vdash t =_0 t'$

• $\{x_1 =_{\epsilon_1} y_1, x_2 =_{\epsilon_2} y_2\} \vdash x_1 \oplus x_2 =_{\max(\epsilon_1,\epsilon_2)} y_1 \oplus y_2$

• $\{x_1 =_{\epsilon_1} y_1, x_2 =_{\epsilon_2} y_2\} \vdash x_1 +_p x_2 =_{p \cdot \epsilon_1 + (1-p) \cdot \epsilon_2} y_1 +_p y_2$

[Mio, V. 2020]

Monad $(\hat{\mathcal{M}}, \hat{\eta}, \hat{\mu})$ in Met $\Leftrightarrow$ Quantitative Equational Theory (Σ, Q)

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Corollary:

$$(\mathcal{C}(X), HK(d)) \cong \left(Terms(X,\Sigma)/_E,\ d_{(\Sigma,Q)}\right)$$

with $d_{(\Sigma,Q)} = (t,t') \mapsto \inf\{\epsilon \mid \emptyset \vdash t =_\epsilon t'\}$

RECAP: ADDING TERMINATION, IN SETS

Monad $(\mathcal{M}, \eta, \mu)$ in Sets $\Leftrightarrow$ Equational Theory (Σ, E)

- Σ a signature
- E a set of equations

Convex sets (possibly empty) of distributions monad $(\mathcal{C}^{\emptyset}, \eta, \mu)$ $\Leftrightarrow$ Equational theory of convex semilattices with bottom $x \oplus \star = x$ and black-hole $x +_p \star = \star$

$\perp$-closed convex sets (possibly empty) of subdistributions monad $(\mathcal{C}^{\downarrow}, \eta, \mu)$ $\Leftrightarrow$ Equational theory of convex semilattices with bottom $x \oplus \star = x$

Lifting to Met: Negative Results

Convex sets
(possibly empty)
of distributions
monad $(\mathcal{C}^{\emptyset}, \eta, \mu)$

$\Leftrightarrow$

Equational theory of convex semilattices
with bottom $x \oplus \star = x$
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Negative results:

- The quantitative equational theory of convex semilattices with bottom and black-hole is trivial
- The multiplication μ of $\mathcal{C}^{\emptyset}$ is not non-expansive $\Rightarrow$ the same monad cannot be lifted to Met

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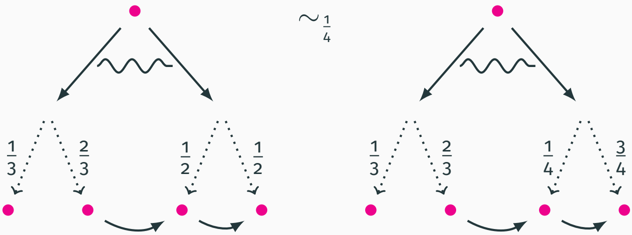
APPLICATION: EQUATIONAL REASONING FOR BISIMULATION METRIC

Processes $P ::= \mathbf{nil} \mid a.P \mid P_1 \overline{\oplus} P_2 \mid P_1 \overline{+}_p P_2$

Semantics $c : \textit{Proc} \to \textit{Terms}(\textit{Proc}, \Sigma)_{/Q}$

with $\Sigma = \star, \oplus, +_p$



RECAP

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RECAP

YES in Met

Convex sets (non-empty) of distributions monad $(\mathcal{C}, \eta, \mu)$ $\Leftrightarrow$ Equational theory of convex semilattices

NO in Met

Convex sets (possibly empty) of distributions monad $(\mathcal{C}^{\emptyset}, \eta, \mu)$ $\Leftrightarrow$ Equational theory of convex semilattices with bottom $x \oplus \star = x$ and black-hole $x +_p \star = \star$

YES in Met

$\perp$-closed convex sets (possibly empty) of subdistributions monad $(\mathcal{C}^{\downarrow}, \eta, \mu)$ $\Leftrightarrow$ Equational theory of convex semilattices with bottom $x \oplus \star = x$

WHAT'S NEXT?

- more process operators, equivalences, metrics
- equational reasoning for trace equivalences and metrics
- recover convex sets of distributions monad in Met, and its presentation, compositionally [Goy, Petrisan 2020]

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Thank you!