MONADS, EQUATIONAL THEORIES AND METRICS
FOR NONDETERMINISTIC AND PROBABILISTIC SYSTEMS

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NONDETERMINISTIC AND PROBABILISTIC SYSTEMS

\begin{center}
\begin{tikzpicture}
  \node (X) at (0,0) {$X$};
  \node (X1) at (-2,-2) {$X_1$};
  \node (X2) at (0,-2) {$X_2$};
  \node (X3) at (2,-2) {$X_3$};
  \node (X4) at (2,-4) {$X_4$};
  \node (X5) at (-2,-4) {$X_5$};
  \node (X6) at (0,-4) {$X_6$};

  \draw[->] (X) to (X1);
  \draw[->] (X) to (X2);
  \draw[->] (X1) to (X5);
  \draw[->] (X2) to (X2);
  \draw[->] (X3) to (X3);
  \draw[->] (X4) to (X4);

  \draw[dotted] (X1) to (X1);
  \draw[dotted] (X2) to (X2);
  \draw[dotted] (X3) to (X3);
  \draw[dotted] (X4) to (X4);

  \node at (-2,-1) {$\frac{1}{2}$};
  \node at (0,-1) {$\frac{1}{2}$};
  \node at (-2,-3) {$\frac{1}{3}$};
  \node at (0,-3) {$\frac{1}{3}$};
  \node at (2,-3) {$\frac{2}{3}$};
  \node at (0,-5) {$\frac{2}{3}$};
\end{tikzpicture}
\end{center}
nondeterminism and probability as computational effects:
monads and equational theories
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monads and equational theories

- equational reasoning in program semantics
nondeterminism and probability as computational effects:
- monads and equational theories
- equational reasoning in program semantics
- program distance $\Rightarrow$ monads on metric spaces and quantitative equational theories
MONADS AND EQUATIONAL THEORIES FOR NONDETERMINISM AND PROBABILITY
Monad \((\mathcal{M}, \eta, \mu)\) in Sets

- functor \(\mathcal{M} : X \mapsto \mathcal{M}(X)\)
- unit \(\eta_X : X \to \mathcal{M}(X)\)
- multiplication \(\mu_X : \mathcal{M}\mathcal{M}(X) \to \mathcal{M}(X)\)
Monad \((\mathcal{M}, \eta, \mu)\)
in Sets

Equational Theory \((\Sigma, E)\)
- \(\Sigma\) a signature
- \(E\) a set of equations

- equations \(t = s\)
- deductive system: equational logic
  \(\{ t = s, s = u \} \vdash t = u\)
- models: algebras \((A, \Sigma^A)\) satisfying the equations
Monads and Equational Theories

Monad \((\mathcal{M}, \eta, \mu)\) in Sets

Equational Theory \(\(\Sigma, E\)\)

- \(\Sigma\) a signature
- \(E\) a set of equations

\((\Sigma, E)\) is a presentation of \((\mathcal{M}, \eta, \mu)\)

The category \(\text{EM}(\mathcal{M})\) of Eilenberg-Moore algebras for \((\mathcal{M}, \eta, \mu)\) is isomorphic to the category \(\text{A}(\Sigma, E)\) of algebras (models) of \((\Sigma, E)\)

Category \(\text{EM}(\mathcal{M})\)
- objects: \((A, \alpha : \mathcal{M}(A) \to A)\) with \(\alpha\) commuting with \(\eta, \mu\)
- arrows: algebra morphisms

Category \(\text{A}(\Sigma, E)\)
- objects: models \((A, \Sigma^A)\) of \((\Sigma, E)\)
- arrows: homomorphisms of \((\Sigma, E)\)-algebras
Monad $(\mathcal{M}, \eta, \mu)$ in Sets

Equational Theory $(\Sigma, E)$
- $\Sigma$ a signature
- $E$ a set of equations

$(\Sigma, E)$ is a presentation of $(\mathcal{M}, \eta, \mu)$

The category $\text{EM}(\mathcal{M})$ of Eilenberg-Moore algebras for $(\mathcal{M}, \eta, \mu)$ is isomorphic to the category $\mathcal{A}(\Sigma, E)$ of algebras (models) of $(\Sigma, E)$

Corollary:
\[ \mathcal{M}(X) \cong \text{Terms}(X, \Sigma)/_E \]
Monad \((\mathcal{M}, \eta, \mu)\) in \text{Sets}

Equational Theory \((\Sigma, E)\)
- \(\Sigma\) a signature
- \(E\) a set of equations

\[\begin{align*}
  c &: X \to \mathcal{P}(X) \\
  c(x) &= \{x_1, x_2\} \\
  c(x_1) &= \{x_1\} \\
  \ldots
\end{align*}\]
**Monad** \((M, \eta, \mu)\) in Sets

Equational Theory \((\Sigma, E)\)
- \(\Sigma\) a signature
- \(E\) a set of equations

Powerset (non-empty) monad \((\mathcal{P}, \eta, \mu)\)
- \(\mathcal{P} : X \mapsto \{S \mid S\ \text{is a non-empty, finite subset of}\ X\}\)
- \(\eta : x \mapsto \{x\}\)
- \(\mu : \{S_1, \ldots, S_n\} \mapsto \bigcup_i S_i\)

Equational theory of semilattices
- \(\Sigma = \text{binary operation} \oplus\)
- axioms of \(E = \)

\[
\begin{align*}
(x \oplus y) \oplus z & \quad \overset{(A)}{=} \quad x \oplus (y \oplus z) \\
 x \oplus y & \quad \overset{(C)}{=} \quad y \oplus x \\
x \oplus x & \quad \overset{(I)}{=} \quad x
\end{align*}
\]
Monad \((\mathcal{M}, \eta, \mu)\) in Sets

Equational Theory \((\Sigma, E)\)
- \(\Sigma\) a signature
- \(E\) a set of equations

Powerset (non-empty) monad \((\mathcal{P}, \eta, \mu)\)
- \(\mathcal{P} : X \mapsto \{S \mid S \text{ is a non-empty, finite subset of } X\}\)
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Equational theory of semilattices
- \(\Sigma = \text{binary operation } \oplus\)
- axioms of \(E = \)
  \[(x \oplus y) \oplus z \overset{(A)}{=} x \oplus (y \oplus z)\]
  \[x \oplus y \overset{(C)}{=} y \oplus x\]
  \[x \oplus x \overset{(I)}{=} x\]

Corollary:
\[\mathcal{P}(X) \cong \text{Terms}(X, \Sigma)/_E\]
EXAMPLE: PROBABILITY

Monad \((\mathcal{M}, \eta, \mu)\)
in Sets

\[ c : X \rightarrow \mathcal{D}(X) \]
\[ c(x) = \frac{1}{2} x_1 + \frac{1}{2} x_2 \]
\[ c(x_1) = 1x_1 \]
...

Equational Theory \((\Sigma, E)\)
- \(\Sigma\) a signature
- \(E\) a set of equations
Monad \((\mathcal{M}, \eta, \mu)\) in Sets

\[ \eta : x \mapsto \{ \Delta \mid \Delta \text{ is a finitely supported probability distribution on } X \} \]

\[ \mu : \sum_i p_i \Delta_i \mapsto \sum_i p_i \cdot \Delta_i \]

Equational Theory \((\Sigma, E)\)
- \(\Sigma\) a signature
- \(E\) a set of equations

Equational theory of convex algebras
- \(\Sigma = \) binary operations \(+_p\) for all \(p \in (0, 1)\)
- Axioms of \(E = \)

\[ (x +_q y) +_p z \quad (A_p) \quad x + pq \left( y + p \frac{(1-q)}{1-pq} z \right) \]

\[ x +_p y \quad (C_p) \quad y +_{1-p} x \]

\[ x +_p x \quad (I_p) \quad x \]
 Monad \((\mathcal{M}, \eta, \mu)\) in Sets

Equational Theory \((\Sigma, E)\)
- \(\Sigma\) a signature
- \(E\) a set of equations

nondeterminism + probability = ?
a transition reaches a set of probability distributions
\[ \{ \frac{1}{2}X_1 + \frac{1}{2}X_2, \frac{1}{3}X_3 + \frac{2}{3}X_4 \} \]

Problem: \( P \circ D \) is not a monad [Varacca, Winskel 2006]
a transition reaches a set of probability distributions
\[ \{ \frac{1}{2}x_1 + \frac{1}{2}x_2, \frac{1}{3}x_3 + \frac{2}{3}x_4 \} \]

Problem: \( \mathcal{P} \circ \mathcal{D} \) is not a monad [Varacca, Winskel 2006]

Solution: use **convex sets of probability distributions**
\[ \{ \frac{1}{2}x_1 + \frac{1}{2}x_2, \ldots, \frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{6}x_3 + \frac{1}{3}x_4, \ldots, \frac{1}{3}x_3 + \frac{2}{3}x_4 \} \]
a transition reaches a set of probability distributions
\[ \left\{ \frac{1}{2}x_1 + \frac{1}{2}x_2, \frac{1}{3}x_3 + \frac{2}{3}x_4 \right\} \]

Problem: \( P \circ D \) is not a monad [Varacca, Winskel 2006]

Solution: use convex sets of probability distributions
\[ \left\{ \frac{1}{2}x_1 + \frac{1}{2}x_2, \ldots, \frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{6}x_3 + \frac{1}{3}x_4, \ldots, \frac{1}{3}x_3 + \frac{2}{3}x_4 \right\} \]

+ accounts for probabilistic schedulers
The monad \((C, \eta, \mu)\) in Sets:

- \(C : X \mapsto \{ S \mid S \text{ is a non-empty, convex-closed, finitely generated set of finitely supported probability distributions over } X \}\)

- \(\eta_X : X \to C(X)\)
  \[\eta_X : x \mapsto \{ 1 \cdot x \}\]

- \(\mu_X : CC(X) \to C(X)\)
  \[\mu_X : \bigcup_{i} \{ \Delta_i \} \mapsto \bigcup_{i} WMS(\Delta_i)\]

with \(WMS : DC(X) \to C(X)\) the weighted Minkowski sum

\[WMS\left(\sum_{i=1}^{n} p_i S_i\right) = \left\{ \sum_{i=1}^{n} p_i \cdot \Delta_i \mid \text{for each } 1 \leq i \leq n, \Delta_i \in S_i \right\}\]

[Jacobs 2008 ...]
THE EQUATIONAL THEORY FOR NONDETERMINISM AND PROBABILITY

Monad \((\mathcal{M}, \eta, \mu)\) in Sets

\[ \text{Equational Theory } (\Sigma, E) \]
- \(\Sigma\) a signature
- \(E\) a set of equations

Convex sets (non-empty) of distributions monad \((\mathcal{C}, \eta, \mu)\)

\[ \text{Equational theory of convex semilattices} \]
- \(\Sigma = \oplus\) and \(+_p\) for all \(p \in (0, 1)\)
- axioms \(E\):
  - axioms of semilattices
  - axioms of convex algebras
  - distributivity axiom \((D)\)
  \[(x \oplus y) +_p z \overset{(D)}{=} (x +_p z) \oplus (y +_p z)\]

[Bonchi, Sokolova, V. 2019 and 2020]
Termination and Equational Reasoning in Program Semantics
Bisimulation equivalence (= coalgebraic behavioral equivalence)

∃ R such that xRy implies:
- if x → x' then y → y' and x'Ry'
- symmetric
c : X → \mathcal{P}(X)

c(x) = \{x_1, x_2\}

c(x_1) = \{x_1\}
...

Bisimulation equivalence (= coalgebraic behavioral equivalence)

\exists R \text{ such that } xRy \text{ implies:}

- if \( x \rightarrow x' \) then \( y \rightarrow y' \) and \( x'Ry' \)
- symmetric

trivial if all states can move \( \Rightarrow \) we need some observable: termination
NONDETERMINISTIC SYSTEMS AND BISIMULATION

\[ c : X \rightarrow \mathcal{P}^\emptyset (X) \]

\[ c(x) = \{ x_1, x_2 \} \]

\[ c(x_1) = \emptyset \quad \text{i.e.,} \quad x_1 \not\xrightarrow{} \]

Bisimulation equivalence (= coalgebraic behavioral equivalence)

\[ \exists \ R \text{ such that } xRy \text{ implies:} \]

- if \( x \rightarrow x' \) then \( y \rightarrow y' \) and \( x'Ry' \)
- symmetric

trivial if all states can move \( \Rightarrow \) we need some observable: termination
Monad $(\mathcal{M}, \eta, \mu)$ in Sets

Equational Theory $(\Sigma, E)$
- $\Sigma$ a signature
- $E$ a set of equations

Powerset (possibly empty) monad $(\mathcal{P}^\emptyset, \eta, \mu)$
- $\mathcal{P}^\emptyset : X \mapsto \{ S \mid S \text{ is a finite subset of } X \}$
- $\eta : x \mapsto \{ x \}$
- $\mu : \{ S_1, \ldots, S_n \} \mapsto \bigcup_i S_i$

Equational theory of semilattices with bottom
- $\Sigma = \star, \oplus$
- axioms of $E =$
  - axioms of semilattices
    - $(x \oplus y) \oplus z \equiv (A) x \oplus (y \oplus z)$
    - $x \oplus y \equiv (C) y \oplus x$
    - $x \oplus x \equiv (I) x$
- bottom axiom $x \oplus \star = x$
Monad \((M, \eta, \mu)\) in Sets

Equational Theory \((\Sigma, E)\)
- \(\Sigma\) a signature
- \(E\) a set of equations

Convex sets (possibly empty) of distributions monad \((C^\emptyset, \eta, \mu)\)

Equational theory of convex semilattices with bottom and black-hole
- \(\Sigma = \ast, \oplus, +_p\) for all \(p \in (0, 1)\)
- axioms of \(E =\)
  - axioms of convex semilattices
  - bottom axiom \(x \oplus \ast = x\)
  - black-hole axiom \(x +_p \ast = \ast\)

[Mio, Sarkis, V. 2021]
APPLICATION: EQUATIONAL REASONING ON PROCESSES

Processes
\[ P ::= \text{nil} \mid a.P \mid P_1 \oplus P_2 \mid P_1 \mp_p P_2 \]

Semantics
\[ c : \text{Proc} \rightarrow \text{Terms}(\text{Proc}, \Sigma) \]

\[ \text{with } \Sigma = *, \oplus, +_p \]

\[
\begin{array}{c}
\text{nil} \quad \frac{\text{nil}}{\text{nil} \rightarrow *} \\
\text{action} \quad \frac{a.P}{a.P \rightarrow P}
\end{array}
\]

\[
\begin{array}{c}
\frac{\oplus}{P_1 \rightarrow t_1 \quad P_2 \rightarrow t_2} \\
\quad \frac{P_1 \oplus P_2}{P_1 \oplus P_2 \rightarrow t_1 \oplus t_2}
\end{array}
\]

\[
\begin{array}{c}
\frac{+_p}{P_1 \rightarrow t_1 \quad P_2 \rightarrow t_2} \\
\quad \frac{P_1 \mp_p P_2}{P_1 \mp_p P_2 \rightarrow t_1 \mp_p t_2}
\end{array}
\]
APPLICATION: EQUATIONAL REASONING ON PROCESSES

Processes

\[
P ::= \text{nil} \mid a.P \mid P_1 \oplus P_2 \mid P_1 \oplus_P P_2
\]

Semantics

\[
c : \text{Proc} \rightarrow \text{Terms}(\text{Proc}, \Sigma)
\]

\[
\text{with } \Sigma = \ast, \oplus, +_p
\]

\[
(a.(\text{nil} \oplus \text{nil}) \oplus_{\frac{1}{3}} a^3.\text{nil}) \oplus (a^2.\text{nil} \oplus_{\frac{1}{2}} a.\text{nil})
\]

\[
((\text{nil} \oplus \text{nil}) +_{\frac{1}{3}} a^2.\text{nil}) \oplus (a.\text{nil} +_{\frac{1}{2}} \text{nil})
\]
APPLICATION: EQUATIONAL REASONING ON PROCESSES

Processes

\[ P ::= \text{nil} \mid a.P \mid P_1 \oplus P_2 \mid P_1 \ominus_p P_2 \]

Semantics

\[ c : 
  \begin{aligned}
  \text{Proc} &\rightarrow \text{Terms}(\text{Proc}, \Sigma) / E \\
  \text{with } \Sigma &= \star, \oplus, +_p 
  \end{aligned}
\]

\[ (a.(\text{nil} \oplus \text{nil}) \ominus_{\frac{1}{3}} a^3.\text{nil}) \oplus (a^2.\text{nil} \ominus_{\frac{1}{2}} a.\text{nil}) \]

\[ [((\text{nil} \oplus \text{nil}) +_{\frac{1}{3}} a^2.\text{nil}) \ominus (a.\text{nil} +_{\frac{1}{2}} \text{nil})] / E \]
Processes \[ P ::= \text{nil} \mid a.P \mid P_1 \oplus P_2 \mid P_1 \mp_p P_2 \]

Semantics \[ c : \text{Proc} \rightarrow \text{Terms(Proc, } \Sigma)_{/E} \simeq C^\emptyset(\text{Proc}) \]

with \( \Sigma = \ast, \oplus, \mp_p \)

\[ (a.(\text{nil } \oplus \text{nil}) \mp_{\frac{1}{3}} a^3.\text{nil}) \oplus (a^2.\text{nil } \mp_{\frac{1}{2}} a.\text{nil}) \]
APPLICATION: EQUATIONAL REASONING ON PROCESSES

Processes
\[ P ::= \text{nil} \mid a.P \mid P_1 \oplus P_2 \mid P_1 \oplus p P_2 \]

Semantics
\[ c : \text{Proc} \rightarrow \text{Terms}(\text{Proc}, \Sigma)_E \simeq C^0(\text{Proc}) \]

with \( \Sigma = \times, \oplus, +_p \)

\[
(a.(\text{nil} \oplus \text{nil}) \oplus \frac{1}{3} \text{a}^3.\text{nil}) \oplus (a^2.\text{nil} \oplus \frac{1}{2} a.a.\text{nil})
\]
**APPLICATION: EQUATIONAL REASONING ON PROCESSES**

Processes
\[ P ::= \text{nil} \mid a.P \mid P_1 \oplus P_2 \mid P_1 +_p P_2 \]

Semantics
\[ c : \text{Proc} \rightarrow \text{Terms}(\text{Proc}, \Sigma)/_E \simeq C^\emptyset(\text{Proc}) \]

with \( \Sigma = *, \oplus, +_p \)

A sound and complete proof system for bisimulation equivalence on processes
\[ P \sim P' \iff P \equiv E P' \]
Processes \( P ::= \textbf{nil} \mid a.P \mid P_1 \oplus P_2 \mid P_1 \odot_p P_2 \)

Semantics \( c : \text{Proc} \rightarrow \text{Terms}(\text{Proc}, \Sigma)_{/E} \cong C^\emptyset(\text{Proc}) \)

with \( \Sigma = \star, \oplus, \odot_p \)

Bottom axiom \( x \oplus \star = x \)

Black-hole axiom \( x \odot_p \star = \star \)
Axioms for Termination: The Issue with Black-Hole

Processes

\[ P ::= \text{nil} \mid a.P \mid P_1 \oplus P_2 \mid P_1 \oplus_p P_2 \]

Semantics

\[ c : \text{Proc} \to \text{Terms}(\text{Proc}, \Sigma)/_E \cong C^\emptyset(\text{Proc}) \]

with \( \Sigma = \ast, \oplus, \oplus_p \)

Bottom axiom \( x \oplus \ast = x \)

Black-hole axiom \( x \oplus_p \ast = \ast \)
AXIOMS FOR TERMINATION: THE ISSUE WITH BLACK-HOLE

Processes

\[ P ::= \text{nil} \mid a.P \mid P_1 \oplus P_2 \mid P_1 
\overset{+}{\rightarrow} P_2 \]

Semantics

\[ c : \text{Proc} \rightarrow \text{Terms}(\text{Proc}, \Sigma)_{/E} \simeq \mathcal{C}^\emptyset(\text{Proc}) \]

with \( \Sigma = \star, \oplus, + \)

\( a.\text{nil} \overset{1}{\rightarrow} \text{nil} \)

\[ \frac{\text{[nil} \overset{1}{\rightarrow} \star]}{E} \]

\[ \simeq \]

\[ \text{nil} \]

\[ \frac{\text{[nil} \overset{1}{\rightarrow} \star]}{E} = \]

\[ \text{[\star]}_{/E} \]

Bottom axiom \( x \oplus \star = x \)

Black-hole axiom \( x \overset{+}{\rightarrow} \star = \star \)
Monad \((\mathcal{M}, \eta, \mu)\) in Sets

Equational Theory \((\Sigma, E)\)
- \(\Sigma\) a signature
- \(E\) a set of equations

\(\perp\)-closed convex sets (possibly empty)
of subdistributions

Monad \((\mathcal{C}^\perp, \eta, \mu)\)

Equational theory of convex semilattices with bottom
- \(\Sigma = \star, \oplus, + p\) for all \(p \in (0, 1)\)
- axioms \(E:\)
  - axioms of convex semilattices
  - bottom axiom \(x \oplus \star = x\)

subdistribution = \(\sum_i p_i x_i\) with \(\sum_i p_i \leq 1\)

\(S\) is \(\perp\)-closed = if \(\sum_i p_i x_i \in S\) then \(\sum_i q_i x_i \in S\) with \(q_i \leq p_i\)

[Mio, Sarkis, V. 2021]
Processes \[ P ::= \text{nil} \mid a.P \mid P_1 \oplus P_2 \mid P_1 \oplus_p P_2 \]

Semantics \[ c : \text{Proc} \to \text{Terms}(\text{Proc}, \Sigma)/E \simeq C^\downarrow(\text{Proc}) \]

with \( \Sigma = \star, \oplus, _p \)

Bottom axiom \[ x \oplus \star = x \]
Processes \[ P ::= \text{nil} \mid a.P \mid P_1 \oplus P_2 \mid P_1 \uplus_p P_2 \]

Semantics \[ c : \text{Proc} \rightarrow \text{Terms}(\text{Proc}, \Sigma)_{/E} \simeq \mathcal{C}^{\downarrow}(\text{Proc}) \]

with \( \Sigma = \star, \oplus, \uplus_p \)

Bottom axiom \[ x \oplus \star = x \]
Processes

\[ P ::= \text{nil} \mid a.P \mid P_1 \oplus P_2 \mid P_1 \oplus_p P_2 \]

Semantics

\[ c : \text{Proc} \rightarrow \text{Terms}(\text{Proc}, \Sigma)_/E \simeq C^\downarrow(\text{Proc}) \]

with \( \Sigma = \star, \oplus, \,+_{p} \)

Bottom axiom

\[ x \oplus \star = x \]
Convex sets (non-empty) of distributions monad \((\mathcal{C}, \eta, \mu)\) ↔ Equational theory of convex semilattices

Convex sets (possibly empty) of distributions monad \((\mathcal{C}^\emptyset, \eta, \mu)\) ↔ Equational theory of convex semilattices with bottom \(x \oplus \star = x\) and black-hole \(x + p \star = \star\)

\(\bot\)-closed convex sets (possibly empty) of subdistributions monad \((\mathcal{C}^\bot, \eta, \mu)\) ↔ Equational theory of convex semilattices with bottom \(x \oplus \star = x\)
MONADS ON METRIC SPACES AND PROGRAM DISTANCES
Category Met of Metric Spaces:

- objects \((X, d)\)
- morphisms \(f : (X, d) \to (Y, d')\) are non-expansive maps between metric spaces

\[
f : X \to Y \quad \text{with} \quad d'(f(x_1), f(x_2)) \leq d(x_1, x_2)
\]

Monad \((M, \eta, \mu)\) in Sets lifted to a monad \((\hat{M}, \hat{\eta}, \hat{\mu})\) in Met, with:

- \(\hat{M} : (X, d) \mapsto (M(X), \text{lift}_M(d))\)
- \(\hat{\eta}_{(X,d)} : (X, d) \to (M(X), \text{lift}_M(d))\) and \(\hat{\mu}_{(X,d)} : (M(M(X), \text{lift}_M \text{lift}_M(d))) \to (M(X), \text{lift}_M(d))\) non-expansive
The monad \((\mathcal{C}, \eta, \mu)\) of convex sets of distributions can be lifted to a monad \((\hat{\mathcal{C}}, \hat{\eta}, \hat{\mu})\) in Met:

\[
\hat{\mathcal{C}} : (X, d) \mapsto (\mathcal{C}(X), \text{HK}(d))
\]

\[
\text{HK}(d) = \text{Hausdorff-Kantorovich lifting of } d
\]
The monad \((\mathcal{C}, \eta, \mu)\) of convex sets of distributions can be lifted to a monad \((\hat{\mathcal{C}}, \hat{\eta}, \hat{\mu})\) in Met:

\[
\hat{C} : (X, d) \mapsto (C(X), HK(d)) \quad HK(d) = \text{Hausdorff-Kantorovich lifting of } d
\]
The monad \((\mathcal{C}, \eta, \mu)\) of convex sets of distributions can be lifted to a monad \((\hat{\mathcal{C}}, \hat{\eta}, \hat{\mu})\) in Met:

\[ \hat{\mathcal{C}} : (X, d) \mapsto (\mathcal{C}(X), HK(d)) \]

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\[ \hat{\mathcal{C}} : (X, d) \mapsto (\mathcal{C}(X), HK(d)) \]

\[ HK(d) = \text{Hausdorff-Kantorovich lifting of } d \]
The monad \((\mathcal{C}, \eta, \mu)\) of convex sets of distributions can be lifted to a monad \((\hat{\mathcal{C}}, \hat{\eta}, \hat{\mu})\) in \(\text{Met}:\)

\[
\hat{\mathcal{C}} : (X, d) \mapsto (\mathcal{C}(X), HK(d))
\]

\[
HK(d) = \text{Hausdorff-Kantorovich lifting of } d
\]
The monad \((C, \eta, \mu)\) of convex sets of distributions can be lifted to a monad \((\hat{C}, \hat{\eta}, \hat{\mu})\) in Met:

\[
\hat{C} : (X, d) \mapsto (C(X), HK(d))
\]

\[
HK(d) = \text{Hausdorff-Kantorovich lifting of } d
\]

\[
\hat{\eta}(x,d) : (X, d) \rightarrow (C(X), HK(d)) \quad \text{and}
\]

\[
\hat{\mu}(x,d) : (CC(X), HK(HK(d))) \rightarrow (C(X), HK(d))
\]

non-expansive
The monad \((\mathcal{C}, \eta, \mu)\) of convex sets of distributions can be lifted to a monad \((\hat{\mathcal{C}}, \hat{\eta}, \hat{\mu})\) in Met:

- \(\hat{\mathcal{C}} : (X, d) \mapsto (\mathcal{C}(X), HK(d))\)

\(HK(d) = \text{Hausdorff-Kantorovich lifting of } d\)

- \(\hat{\eta}_{(X,d)} : (X, d) \to (\mathcal{C}(X), HK(d))\) and
- \(\hat{\mu}_{(X,d)} : (\mathcal{C}(X), HK(HK(d))) \to (\mathcal{C}(X), HK(d))\)

non-expansive

[Mio, V. 2020]
Monad \((\hat{\mathcal{M}}, \hat{\eta}, \hat{\mu})\) in Met

\[ \begin{align*} & \Sigma \text{ a signature} \\
& Q \text{ a set of quantitative inferences} \end{align*} \]
**MONADS ON METRIC SPACES AND QUANTITATIVE EQUATIONAL THEORIES**

Monad $(\mathcal{M}, \hat{\eta}, \hat{\mu})$ in Met

Quantitative Equational Theory $(\Sigma, Q)$

- $\Sigma$ a signature
- $Q$ a set of quantitative inferences

- quantitative equations $t =_{\epsilon} t'$
- quantitative inferences $\{t_i =_{\epsilon_i} s_i\}_{i \in I} \vdash t =_{\epsilon} s$

- quantitative deduction system
  - (Reflexivity) $\emptyset \vdash t =_{o} t$
  - (Symmetry) $\{t =_{\epsilon} s\} \vdash s =_{\epsilon} t$
  - (Triangular) $\{t =_{\epsilon_1} u, u =_{\epsilon_2} s\} \vdash t =_{\epsilon_1 + \epsilon_2} s$

- models: quantitative algebras $(A, \Sigma^A, d)$ satisfying the quantitative inferences

[Mardare, Panangaden, Plotkin 2016]
Monad \((\hat{\mathcal{M}}, \hat{\eta}, \hat{\mu})\) in Met

Quantitative Equational Theory \((\Sigma, Q)\)
- \(\Sigma\) a signature
- \(Q\) a set of quantitative inferences

\((\Sigma, Q)\) is a presentation of \((\hat{\mathcal{M}}, \hat{\eta}, \hat{\mu})\)

The category \(\text{EM}(\hat{\mathcal{M}})\) of Eilenberg-Moore algebras for \((\hat{\mathcal{M}}, \hat{\eta}, \hat{\mu})\) is isomorphic to the category \(\text{A}(\Sigma, E)\) of quantitative algebras (models) of \((\Sigma, Q)\)
**THE QUANTITATIVE EQUATIONAL THEORY OF CONVEX SEMILATTICES**

Monad $(\mathcal{M}, \hat{\eta}, \hat{\mu})$ in Met

Quantitative Equational Theory $(\Sigma, Q)$
- $\Sigma$ a signature
- $Q$ a set of quantitative inferences

Convex sets (non-empty) of distributions monad $(\mathcal{C}, \hat{\eta}, \hat{\mu})$ in Met

Quantitative equational theory of convex semilattices
- $\Sigma = \oplus$ and $+p$ for all $p \in (0, 1)$
- quantitative inferences $Q =$
  - axioms of convex semilattices, with $t = t'$ becoming $\emptyset \vdash t =_o t'$
  - $\{x_1 =_{\epsilon_1} y_1, x_2 =_{\epsilon_2} y_2\} \vdash x_1 \oplus x_2 =_{\max(\epsilon_1, \epsilon_2)} y_1 \oplus y_2$
  - $\{x_1 =_{\epsilon_1} y_1, x_2 =_{\epsilon_2} y_2\} \vdash x_1 +_p x_2 =_{p \cdot \epsilon_1 + (1-p) \cdot \epsilon_2} y_1 +_p y_2$

[Mio, V. 2020]
The Quantitative Equational Theory of Convex Semilattices

Monad \((\hat{\mathcal{M}}, \hat{\eta}, \hat{\mu})\) in Met

Quantitative Equational Theory \((\Sigma, Q)\)
- \(\Sigma\) a signature
- \(Q\) a set of quantitative inferences

Corollary:

\[
(C(X), HK(d)) \cong (\text{Terms}(X, \Sigma)/E, d(\Sigma, Q))
\]

with \(d(\Sigma, Q) = (t, t') \mapsto \inf\{\epsilon \mid \emptyset \vdash t =_{\epsilon} t'\}\)
RECAP: ADDING TERMINATION, IN SETS

Monad \((M, \eta, \mu)\) in Sets

Equational Theory \((\Sigma, E)\)
- \(\Sigma\) a signature
- \(E\) a set of equations

Convex sets (possibly empty) of distributions monad \((C^\emptyset, \eta, \mu)\)

Equational theory of convex semilattices with bottom \(x \oplus \star = x\) and black-hole \(x + p \star = \star\)

\(\perp\)-closed convex sets (possibly empty) of subdistributions monad \((C^\perp, \eta, \mu)\)

Equational theory of convex semilattices with bottom \(x \oplus \star = x\)
Convex sets
(possibly empty)
of distributions
monad \((C^\emptyset, \eta, \mu)\)

Equational theory of convex semilattices
with bottom \(x \oplus \star = x\)
and black-hole \(x +_p \star = \star\)

Negative results:

- The quantitative equational theory of convex semilattices with bottom and black-hole is trivial.
- The multiplication \(\mu\) of \(C^\emptyset\) is not non-expansive \(\Rightarrow\) the same monad cannot be lifted to Met.
LIFTING TO MET: NEGATIVE RESULTS

Convex sets (possibly empty) of distributions monad \((C^\emptyset, \eta, \mu)\)

\[ x \oplus \star = x \]

\[ x +_p \star = \star \]

Equational theory of convex semilattices with bottom and black-hole

Negative results:

- The quantitative equational theory of convex semilattices with bottom and black-hole is trivial
- The multiplication \(\mu\) of \(C^\emptyset\) is not non-expansive \(\Rightarrow\) the same monad cannot be lifted to Met

\(\perp\)-closed convex sets (possibly empty) of subdistributions monad \((C^\perp, \eta, \mu)\)

Equational theory of convex semilattices with bottom

\[ x \oplus \star = x \]
Monad \((\mathcal{M}, \hat{\eta}, \hat{\mu})\) in Met

\[
\begin{align*}
\vdash \text{Quantitative Equational Theory (}\Sigma, Q\text{)} & \\
\Sigma & \text{a signature} \\
Q & \text{a set of quantitative inferences}
\end{align*}
\]

\(\bot\)-closed convex sets (possibly empty) of subdistributions
monad \((\mathcal{C}^\bot, \hat{\eta}, \hat{\mu})\) in Met

\[
\begin{align*}
\vdash \text{Quantitative equational theory of convex semilattices with bottom } & x \oplus \star = x
\end{align*}
\]
Processes $P ::= \text{nil} \mid a.P \mid P_1 \oplus P_2 \mid P_1 \vdash_p P_2$

Semantics $c : \text{Proc} \rightarrow \text{Terms}(\text{Proc}, \Sigma)/Q$

with $\Sigma = \ast, \oplus, +_p$
Convex sets (non-empty) of distributions monad $(\mathcal{C}, \eta, \mu)$ $\iff$ Equational theory of convex semilattices

Convex sets (possibly empty) of distributions monad $(\mathcal{C}^\emptyset, \eta, \mu)$ $\iff$ Equational theory of convex semilattices with bottom $x \oplus \ast = x$ and black-hole $x \oplus_p \ast = \ast$

$\perp$-closed convex sets (possibly empty) of subdistributions monad $(\mathcal{C}^\perp, \eta, \mu)$ $\iff$ Equational theory of convex semilattices with bottom $x \oplus \ast = x$
Convex sets (non-empty) of distributions monad \((C, \eta, \mu)\)

\[\iff\]

Equational theory of convex semilattices

Convex sets (possibly empty) of distributions monad \((C^\emptyset, \eta, \mu)\)

\[\iff\]

Equational theory of convex semilattices with bottom \(x \oplus \star = x\)

and black-hole \(x +_p \star = \star\)

\[\iff\]

Equational theory of convex semilattices with bottom \(x \oplus \star = x\)

\(\perp\)-closed convex sets (possibly empty) of subdistributions monad \((C^\perp, \eta, \mu)\)

YES in Met

NO in Met

YES in Met
WHAT’S NEXT?

- more process operators, equivalences, metrics
- equational reasoning for trace equivalences and metrics
- recover convex sets of distributions monad in Met, and its presentation, compositionally [Goy, Petrisan 2020]
- more process operators, equivalences, metrics
- equational reasoning for trace equivalences and metrics
- recover convex sets of distributions monad in Met, and its presentation, compositionally [Goy, Petrisan 2020]