Functional interpretations and applications

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Rencontres mensuelles “CHoCoLa”

January 20, 2022
Overview

Amuse-bouche

BFI

First course: functional interpretations for NSA
   Nonstandard analysis in proof theory
   Nonstandard Realizability
   Nonstandard Intuitionistic functional interpretation

Second course: a parametrised interpretation
   Parametrised interpretations of AL
   Parametrised interpretations of IL
   Instances

Dessert: realizability with stateful computations for NSA
Outline

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In fact, there exist explicit examples ("Specker sequences") of sequences of computable reals with no computable limit and thus with no computable rate of convergence.
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The next best thing is then what Terence Tao called a *rate of metastability*, i.e., a bound on the $N$ in the statement
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\[
\forall \varepsilon > 0 \forall f : \mathbb{N} \to \mathbb{N} \exists N \forall i, j \in [N, N + f(N)](\|x_i - x_j\| \leq \varepsilon)
\]
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Metastability

\[
\forall k \in \mathbb{N} \forall f : \mathbb{N} \to \mathbb{N} \exists N \forall i, j \in [N, N + f(N)] \left( \|x_i - x_j\| \leq \frac{1}{k+1} \right)
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which is a Herbrandization of the Cauchy property of a sequence.
Proof mining

Proof mining program → analyses of mathematical proofs with the help of proof theoretic techniques, including functional interpretations, in search of concrete new information: effective bounds, algorithms, weakening of premisses, ...
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- Allow to obtain explicit bounds
A very short (and biased) history of proof mining

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Functional interpretations

A functional interpretation is a mapping $f : S \rightarrow T$ such that a formula $A$ (in classical logic) is mapped to a formula

$$A^f \equiv \forall x \exists y \ A_f(x, y)$$

such that theorems of $S$ are mapped to theorems of $T$, i.e.

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Moreover, \( f \) provides a witness for the existential quantifier (term).

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Functional interpretations allow for the extraction of the (hidden) computational content (captured by $t$) in the proof of the theorem.
Interpretations with different flavours

- Kleene (numerical realizability) (1952)
- Gödel (Dialectica) (1958)
- Kreisel (modified realizability) (1959)
- Diller and Nahm (variant to avoid the contraction problem) (1974)
- Stein (family of interpretations) (1979)
- Ferreira and Oliva (bounded functional interpretation) (2005)
- Van den Berg, Briseid and Safarik (Herbrandized) (2012)
- ...
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We use the Bounded Functional Interpretation (BFI) and its characteristic principles, enriched with a new base type for elements of the space and the (universal) axioms for the Hilbert space.
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- Completely new translation of formulas
- Independence on bounded parameters is made explicit (via the interpretation itself)
Majorizability

Let $\text{PA}^\omega$ be Peano Arithmetic in all finite types. Types are defined inductively as follows

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**Definition**

- The **Howard-Bezem strong majorizability** $\leq^*_\sigma$ is defined by:
  - $s \leq^*_0 t :\equiv s \leq_0 t$;
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- The Howard-Bezem strong majorizability $\leq^*_\sigma$ is defined by:
  - $s \leq^*_0 t \equiv s \leq_0 t$;
  - $s \leq^*_{\rho \rightarrow \sigma} t \equiv \forall v \forall u \leq^*_\rho v (su \leq^*_\sigma tv \land tu \leq^*_\sigma tv)$.
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- $\leq^*_\sigma$ is not reflexive! We say that $x^\sigma$ is monotone if and only if $x \leq^*_\sigma x$. 
Majorizability

Proposition

1. $\text{PA}_{\leq^*} \vdash x \leq^*_\sigma y \rightarrow y \leq^*_\sigma y$;
2. $\text{PA}_{\leq^*} \vdash x \leq^*_\sigma y \land y \leq^*_\sigma z \rightarrow x \leq^*_\sigma z$.

Theorem (Howard’s majorizability theorem)

*For all closed terms $t^\sigma$ of $\text{PA}_{\leq^*}^{\omega}$, there is a closed term $s^\sigma$ of $\text{PA}_{\leq^*}^{\omega}$ such that $\text{PA}_{\leq^*} \vdash t \leq^*_\sigma s$.**
Quantifiers

The usual (unbounded quantifiers) $\forall x A(x)$ and $\exists x A(x)$.

Formulas that don't contain unbounded quantifiers are called bounded formulas.
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Formulas that don’t contain unbounded quantifiers are called bounded formulas.
Assign to each formula $A$ of $\text{PA}^\omega_{\leq \ast}$ the formulas $A^f$ and $A_f(a; b)$ of $\text{PA}^\omega_{\leq \ast}$ such that $A^f \equiv \exists a \exists b A_f(a; b)$ according to the following clauses.
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2. $(A \lor B)^f :\equiv \exists a, c \exists b, d (A_f(a; b) \lor B_f(c; d))$;
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3. $(\neg A)^f \equiv \exists h \exists a \exists a' \leq^* a \neg A_f(a'; ha');$
Bounded functional interpretation (Ferreira and Oliva)

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4. $(\forall x A(x))^f \equiv \exists e \exists a \exists b \forall x \leq^* e A_f(x, a; b);$
Assign to each formula $A$ of $\text{PA}_\omega^\ast$ the formulas $A^f$ and $A_f(a; b)$ of $\text{PA}_\omega^\ast$ such that $A^f \equiv \tilde{\forall} a \tilde{\exists} b A_f(a; b)$ according to the following clauses.

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5. $(\forall x \leq^* t A(x))^f : \equiv \tilde{\forall} a \tilde{\exists} b \forall x \leq^* t A_f(x, a; b).$
Caracteristic Principles

**Definition**

1. \((mAC_{bd}) \equiv \forall x \exists y \ A_{bd}(x, y) \rightarrow \exists f \forall x \exists y \leq* fx A_{bd}(x, y);\)
Caracteristic Principles

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1. \((\text{mAC}_{\omega}^{bd}) \equiv \forall x \exists y \ A_{bd}(x, y) \rightarrow \exists f \forall x \exists y \leq^* fx \ A_{bd}(x, y)\);

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3. \((\text{MAJ}^\omega) \equiv \forall x \exists y(x \leq^* y)\).
Soundness

Theorem (soundness theorem of \( f \))

For all formulas \( A \) of \( \text{PA}_{\leq}^\omega \), if

\[
\text{PA}_{\leq}^\omega + P \vdash A,
\]

then there are closed monotone terms \( t \) of appropriate types such that

\[
\text{PA}_{\leq}^\omega \vdash \forall a \exists b \leq^* ta A_f(a; b).
\]

Abbreviation

\( P := \text{MAC}_{bd}^\omega + \text{Coll}_{bd}^\omega + \text{MAJ}^\omega. \)
Characterization

**Theorem (characterization theorem of $f$)**

*For all formulas $A$ of $\text{PA}_{\leq \ast}^\omega$, we have*

$$\text{PA}_{\leq \ast}^\omega + P \vdash A \leftrightarrow A^f.$$

**Abbreviation**

$P := \text{mAC}_{\text{bd}}^\omega + \text{Coll}_{\text{bd}}^\omega + \text{MAJ}^\omega.$
From arithmetic to Hilbert spaces

We add:

▶ a new base type $H$ for objects in an abstract Hilbert space and extend the notion of majorizability in an appropriate way.

▶ axioms characterizing the abstract space and all the required new constants.

▶ modulus (of convergence, of "Cauchyness", of asymptotic regularity, of metastability, etc.) witnessing problematic existential quantifiers.

As long as the new constants are majorizable and the new axioms are universal the proof of the Soundness theorem can be extended to this new theory.
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As long as the new constants are majorizable and the new axioms are universal the proof of the Soundness theorem can be extended to this new theory.
An example: Browder’s theorem

Theorem (Browder 1967)

Let $H$ be an Hilbert space and $U : H \to H$ a non-expansive map. Suppose that $C$ is a convex, closed and bounded subset of $H$, $0 \in C$ and that $U$ maps $C$ into $C$. For every $n \in \mathbb{N}$, let $U_n : H \to H$ the strict contraction $U_n(x) = (1 - \frac{1}{n+1})U(x)$ and let $u_n$ the unique fixed point of $U_n$. Then the sequence $(u_n)$ strongly converges for a fixed point $u \in C$ of $U$. 
A quantitative version of Browder’s theorem

Theorem (Kohlenbach 2011; Ferreira, Leustean, Pinto 2019)

For all \( k \in \mathbb{N} \) and function \( f : \mathbb{N} \to \mathbb{N} \),

\[
\exists n \leq \phi(k, f) \forall i, j \in [n, n + fn] \left( \|u_i - u_j\| \leq \frac{1}{2^k} \right).
\]
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\]

For \( f \) increasing one obtains the following rate of convergence

\[
\phi(k, f) := 2^{g_k(r)(0)+4+2d}
\]

where

- \( d \) is an upper bound of the diameter of \( C \).
- \( g_k(n) := 2k + d + 5 + \lceil \log_2(2^{2n+4+2d}) + f(2^{2n+4+2d}) + 1 \rceil \).
- \( r := 2^{2k+4d+9} \).
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Some (arithmetical) intuitions

- Conservative extension
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- Nonstandard naturals are "big"
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- Overspill and Underspill
The simplest example: ENA

Extend the language of mathematics (e.g. ZFC) with a new (undefined) predicate $st$. 
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Extend the language of mathematics (e.g. ZFC) with a new (undefined) predicate \( st \)

**Internal** formulas = ”Without \( st \)”.
**External** formulas = ”With \( st \)”.
The axioms of ENA

Axiom

\[ \text{st}(0) \]
The axioms of ENA

Axiom

- \( \text{st}(0) \)
- \( \forall n \in \mathbb{N} (\text{st}(n) \Rightarrow \text{st}(n + 1)) \)
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- \( \text{st}(0) \)
- \( \forall n \in \mathbb{N} (\text{st}(n) \Rightarrow \text{st}(n + 1)) \)
- \( \exists \omega \in \mathbb{N} (\neg \text{st}(\omega)) \)
The axioms of ENA

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For each external formula \( \Phi \)

| ▶ \( (\Phi(0) \land \forall^{st} n(\Phi(n) \Rightarrow \Phi(n + 1))) \Rightarrow \forall^{st} n \Phi(n) \) |
The axioms of ENA

Axiom

- $\text{st}(0)$
- $\forall n \in \mathbb{N}(\text{st}(n) \Rightarrow \text{st}(n + 1))$
- $\exists \omega \in \mathbb{N}(\neg \text{st}(\omega))$

For each external formula $\Phi$

- $(\Phi(0) \land \forall^{\text{st}} n (\Phi(n) \Rightarrow \Phi(n + 1))) \Rightarrow \forall^{\text{st}} n \Phi(n)$

$\forall^{\text{st}} n \Phi(n)$ abbreviates $\forall n (\text{st}(n) \Rightarrow \Phi(n))$. 
How to be nonstandard?

- **Model theory**: Compactness theorem, ultrafilters, ultralimits, superstructures,... (Robinson, Luxemburg, Keisler, ...)
- **Set theory**: IST, HST,... Language \{\in, \text{st}\} (Nelson, Hrbacek, Kanovei, Reeken, ...)
- **Algebraic**: (Benci, Di Nasso and Forti, D. and Van den Berg)
Functional interpretations using NSA

- Pioneer works by Moerdijk, Palmgren and Avigad
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- “A functional interpretation of nonstandard arithmetic” (Van den Berg et al.)
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- “Intuitionistic nonstandard bounded interpretations” (D., Gaspar)
- “Realizability with stateful computations for NSA” (D., Miquey)

Most works are inspired by Nelson’s IST
Internal set theory

- **Transfer:** \( A(x) \) internal

\[ \forall^{\text{st}} x. A(x) \implies \forall x. A(x) \]

- **Idealization:** \( R(x, y) \) internal relation

\[ \forall^{\text{stfin}} z. \exists y. \forall x \in z. R(x, y) \implies \exists y. \forall^{\text{st}} x. R(x, y) \]

- **Standardization:** For any \( C(x) \)

\[ \forall^{\text{st}} B. \exists^{\text{st}} A. \forall^{\text{st}} z. (z \in A \iff z \in B \land C(z)) \]
Enrich the language and the axioms of $E-HA^\omega_{st}$ as follows.

- $st^\sigma(t^\sigma)$ (for each finite type $\sigma$).
Enrich the language and the axioms of $\text{E-HA}^\omega_{st}$ as follows.

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- **Standardness axioms:**
Enrich the language and the axioms of $\text{E-HA}^\omega_{\text{st}}$ as follows.

- $\text{st}^\sigma(t^\sigma)$ (for each finite type $\sigma$).
- **Standardness axioms:**
  - $x =_\sigma y \land \text{st}^\sigma(x) \rightarrow \text{st}^\sigma(y)$;
E-\(HA^\omega_{st}\)

Enrich the language and the axioms of \(E-\text{HA}^\omega\) as follows.

- \(\text{st}^\sigma(t^\sigma)\) (for each finite type \(\sigma\)).
- **Standardness axioms:**
  - \(x =_\sigma y \land \text{st}^\sigma(x) \rightarrow \text{st}^\sigma(y)\);
  - \(\text{st}^\sigma(y) \land x \leq^*_\sigma y \rightarrow \text{st}^\sigma(x)\);
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- $st^\sigma(t^\sigma)$ (for each finite type $\sigma$).

- **Standardness axioms:**
  - $x =_\sigma y \land st^\sigma(x) \rightarrow st^\sigma(y)$;
  - $st^\sigma(y) \land x \leq^*_\sigma y \rightarrow st^\sigma(x)$;
  - $st^\sigma(t)$ for each closed term $t$;
Enrich the language and the axioms of $\text{E-HA}^\omega$ as follows.

- $\text{st}^\sigma(t^\sigma)$ (for each finite type $\sigma$).

- **Standardness axioms:**
  - $x =_\sigma y \land \text{st}^\sigma(x) \rightarrow \text{st}^\sigma(y)$;
  - $\text{st}^\sigma(y) \land x \leq^*_\sigma y \rightarrow \text{st}^\sigma(x)$;
  - $\text{st}^\sigma(t)$ for each closed term $t$;
  - $\text{st}^\sigma \rightarrow^\tau(x) \land \text{st}^\sigma(y) \rightarrow \text{st}^\tau(xy)$;
Enrich the language and the axioms of $\text{E-HA}^\omega_{\text{st}}$ as follows.

- $\text{st}^\sigma(t^\sigma)$ (for each finite type $\sigma$).

- **Standardness axioms:**
  - $\forall x, y \ (x =_\sigma y \land \text{st}^\sigma(x) \rightarrow \text{st}^\sigma(y))$;
  - $\forall x, y \ (\text{st}^\sigma(y) \land x \leq^*_\sigma y \rightarrow \text{st}^\sigma(x))$;
  - $\text{st}^\sigma(t)$ for each closed term $t$;
  - $\text{st}^{\sigma \rightarrow \tau}(x) \land \text{st}^\sigma(y) \rightarrow \text{st}^\tau(xy)$;

- **External induction rule:**

$$
\Phi(0) \quad \forall x^0 \ (\text{st}^0(x) \rightarrow (\Phi(x) \rightarrow \Phi(x + 1)))
\quad \forall x^0 \ (\text{st}^0(x) \rightarrow \Phi(x))
$$
Some abbreviations

- $\forall x \varphi(x)$ abbreviates $\forall x (x \leq^* x \rightarrow \varphi(x))$.
- $\exists x \varphi(x)$ abbreviates $\exists x (x \leq^* x \land \varphi(x))$.
- $\forall^{st} x \varphi(x)$ abbreviates $\forall x (st(x) \rightarrow \varphi(x))$.
- $\exists^{st} x \varphi(x)$ abbreviates $\exists x (st(x) \land \varphi(x))$.
- $\ldots$
Nonstandard bounded modified realizability
(jww J. Gaspar)

Assign to each formula $\Phi$ of $E$-$\text{HA}^\omega_{st}$ the formulas $\Phi^b$ and $\Phi^b(a)$ of $E$-$\text{HA}^\omega_{st}$ such that $\Phi^b \equiv \exists^{st} a \Phi^b(a)$ according to the following clauses:

1. $\Phi^b \equiv [\Phi]$ for internal atomic formulas $\Phi$;
2. $st(t)^b \equiv \exists^{st} a [t \leq^* a]$;
Assign to each formula $\Phi$ of $E$-$\text{HA}^\omega_{st}$ the formulas $\Phi^b$ and $\Phi_b(a)$ of $E$-$\text{HA}^\omega_{st}$ such that $\Phi^b \equiv \exists^\text{st} a \Phi_b(a)$ according to the following clauses:

1. $\Phi^b :\equiv [\Phi]$ for internal atomic formulas $\Phi$;
2. $\text{st}(t)^b :\equiv \exists^\text{st} a [t \leq^* a]$;

If $\Phi^b \equiv \exists^\text{st} a \Phi_b(a)$ and $\Psi^b \equiv \exists^\text{st} b \Psi_b(b)$, then:
Assign to each formula $\Phi$ of $\text{E-HA}^\omega_{st}$ the formulas $\Phi^b$ and $\Phi^\omega_b(a)$ of $\text{E-HA}^\omega_{st}$ such that $\Phi^b \equiv \tilde{\exists}^\omega_{st} a \Phi^\omega_b(a)$ according to the following clauses:

1. $\Phi^b \equiv [\Phi]$ for internal atomic formulas $\Phi$;
2. $\text{st}(t)^b \equiv \tilde{\exists}^\omega_{st} a [t \leq^\star a]$;

If $\Phi^b \equiv \tilde{\exists}^\omega_{st} a \Phi^\omega_b(a)$ and $\Psi^b \equiv \tilde{\exists}^\omega_{st} b \Psi^\omega_b(b)$, then:

3. $(\Phi \land \Psi)^b \equiv \tilde{\exists}^\omega_{st} a, b [\Phi^\omega_b(a) \land \Psi^\omega_b(b)]$;
4. $(\Phi \lor \Psi)^b \equiv \tilde{\exists}^\omega_{st} a, b [\Phi^\omega_b(a) \lor \Psi^\omega_b(b)]$;
5. $(\Phi \rightarrow \Psi)^b \equiv \tilde{\exists}^\omega_{st} B [\tilde{\forall}^\omega_{st} a (\Phi^\omega_b(a) \rightarrow \Psi^\omega_b(Ba))]$;
6. $(\forall x \Phi)^b \equiv \tilde{\exists}^\omega_{st} a [\forall x \Phi^\omega_b(a)]$;
7. $(\exists x \Phi)^b \equiv \tilde{\exists}^\omega_{st} a [\exists x \Phi^\omega_b(a)]$. 
Lemma (monotonicity of $b$)

For all formulas $\Phi$ of $E-HA_{st}^\omega$, we have

$$E-HA_{st}^\omega \vdash \Phi_b(a) \land a \leq^* c \rightarrow \Phi_b(c).$$
\textbf{Definition}

We say that a formula of $\text{E-HA}_\omega^{\text{st}}$ is $\exists^\text{st}$-free if and only if it is built:

1. from atomic internal formulas $s =_0 t$;
2. by conjunctions $\land$;
3. by disjunctions $\lor$;
4. by implications $\rightarrow$;
5. by quantifications $\forall$ and $\exists$ (so also $\tilde{\forall}$ and $\tilde{\exists}$);
6. by monotone standard universal quantifications $\forall^{\text{st}}$ (but, of course, not $\exists^{\text{st}}$).
Lemma

For all $\exists^*_{st}$-free formulas $\Phi_{\exists^*_{st}}$ of $E-HA^\omega_{st}$, we have

- $(\Phi_{\exists^*_{st}})^b \equiv (\Phi_{\exists^*_{st}})_b(a)$;
- $E-HA^\omega_{st} \vdash (\Phi_{\exists^*_{st}})_b \leftrightarrow \Phi_{\exists^*_{st}}$. 
Lemma

- For all \( \exists^{st} \)-free formulas \( \Phi_{\exists^{st}} \) of \( E-HA_{st}^{\omega} \), we have
  - \( (\Phi_{\exists^{st}})^b \equiv (\Phi_{\exists^{st}})_b(a) \);
  - \( E-HA_{st}^{\omega} \vdash (\Phi_{\exists^{st}})_b \iff \Phi_{\exists^{st}} \).

- For all formulas \( \Phi \) of \( E-HA_{st}^{\omega} \), the formula \( \Phi_b(a) \) is \( \exists^{st} \)-free.
Caracteristic Principles

**Definition**

- \( \text{mAC}^\omega \equiv \forall^{\text{st}} x \exists^{\text{st}} y \Phi \rightarrow \exists^{\text{st}} Y \forall^{\text{st}} x \exists^{\text{st}} y \leq^* Yx \Phi \);
- \( \text{R}^\omega \equiv \forall x \exists^{\text{st}} y \Phi \rightarrow \exists^{\text{st}} z \forall x \exists y \leq^* z \Phi \);
- \( \text{IP}^{\omega}_{\#_{\text{st}}} \equiv (\Phi^{\#_{\text{st}}} \rightarrow \exists^{\text{st}} x \Psi) \rightarrow \exists^{\text{st}} y (\Phi^{\#_{\text{st}}} \rightarrow \exists x \leq^* y \Psi) \);
- \( \text{MAJ}^\omega \equiv \forall^{\text{st}} x \exists^{\text{st}} y (x \leq^* y) \).

**Proposition**
The principle \( \text{R}^\omega \) implies the principle \( \text{MAJ}^\omega \), that is \( \varepsilon_{\text{E-}} \text{HA}^\omega \) proves all instances of \( \text{MAJ}^\omega \).
Caracteristic Principles

Definition

- \( mAC^\omega \equiv \exists^* x \exists^* y \Phi \rightarrow \exists^* Y \exists^* x \exists^* y \leq^* Yx \Phi; \)
- \( R^\omega \equiv \forall x \exists^* y \Phi \rightarrow \exists^* z \forall x \exists y \leq^* z \Phi; \)
- \( IP_{\#st}^\omega \equiv (\Phi_{\#st} \rightarrow \exists^* x \Psi) \rightarrow \exists^* y (\Phi_{\#st} \rightarrow \exists x \leq^* y \Psi); \)
- \( MAJ^\omega \equiv \forall^* x \exists^* y (x \leq^* y). \)

Proposition

*The principle \( R^\omega \) implies the principle \( MAJ^\omega \), that is \( E-HA_{st}^\omega + R^\omega \) proves all instances of \( MAJ^\omega \).*
Soundness

Theorem (soundness theorem of \( b \))

For all formulas \( \Phi \) of \( E\text{-HA}_{st}^\omega \), if

\[
E\text{-HA}_{st}^\omega + P \vdash \Phi,
\]

then there are closed monotone terms \( t \) of appropriate types such that

\[
E\text{-HA}_{st}^\omega \vdash \Phi_b(t).
\]

Abbreviation

\[
P := E\text{-HA}_{st}^\omega + mAC^\omega + R^\omega + IP_{\frac{\omega}{2}}^{\not\exists_{st}} + MAJ^\omega.
\]
Characterization

**Theorem (Characterization theorem of b)**

For all formulas $\Phi$ of $E$-$HA^{\omega}_{st}$, we have

$$E$-$HA^{\omega}_{st} + P \vdash \Phi \leftrightarrow \Phi^b.$$ 

**Abbreviation**

$P := E$-$HA^{\omega}_{st} + mAC^{\omega} + R^{\omega} + IP^{\omega}_{\#st} + MAJ^{\omega}.$
Intuitionistic nonstandard bounded functional interpretation

Assign to each formula $\Phi$ of $\text{E-HA}_{st}^\omega$ the formulas $\Phi^B$ and $\Phi_B(a; b)$ of $\text{E-HA}_{st}^\omega$ such that $\Phi^B \equiv \exists^\st a \forall^\st b \Phi_B(a; b)$ according to the following clauses.

1. $\Phi^B :\equiv [\Phi]$ for internal atomic formulas $\Phi$;
2. $\text{st}(t)^B :\equiv \exists^\st a [t \leq^* a]$. 
Intuitionistic nonstandard bounded functional interpretation

Assign to each formula $\Phi$ of $E\text{-}HA^\omega_{st}$ the formulas $\Phi^B$ and $\Phi_B(a; b)$ of $E\text{-}HA^\omega_{st}$ such that $\Phi^B \equiv \exists^{st} a \forall^{st} b \Phi_B(a; b)$ according to the following clauses.

1. $\Phi^B \equiv [\Phi]$ for internal atomic formulas $\Phi$;
2. $st(t)^B \equiv \exists^{st} a [t \leq^* a]$.

If $\Phi^B \equiv \exists^{st} a \forall^{st} b \Phi_B(a; b)$ and $\Psi^B \equiv \exists^{st} c \forall^{st} d \Psi_B(c; d)$ then:
Assign to each formula Φ of E-HA_{st} the formulas Φ^B and Φ_B(a; b) of E-HA_{st} such that Φ^B ≡ ⩾_st a ⩾_st b Φ_B(a; b) according to the following clauses.

1. Φ^B :≡ [Φ] for internal atomic formulas Φ;
2. st(t)^B :≡ ⩾_st a [t ≤^* a].

If Φ^B ≡ ⩾_st a ⩾_st b Φ_B(a; b) and Ψ^B ≡ ⩾_st c ⩾_st d Ψ_B(c; d) then:

3. (Φ ∧ Ψ)^B :≡ ⩾_st a, c ⩾_st b, d [Φ_B(a; b) ∧ Ψ_B(c; d)];
4. (Φ ∨ Ψ)^B :≡ ⩾_st a, c ⩾_st e, f
   [⩾_st b ≤^* e Φ_B(a; b) ∨ ⩾_st d ≤^* f Ψ_B(c; d)];
5. (Φ → Ψ)^B :≡ ⩾_st C, B ⩾_st a, d
   [⩾_st b ≤^* Bsd Φ_B(a; b) → Ψ_B(Ca; d)];
6. (∀x Φ)^B :≡ ⩾_st a ⩾_st b [∀x Φ_B(a; b)];
7. (∃x Φ)^B :≡ ⩾_st a ⩾_st c [∃x ⩾_st b ≤^* c Φ_B(a; b)].
Monotonicity

**Lemma (monotonicity of B)**

*For all formulas $\Phi$ of $E$-HA$^\omega_{st}$, we have*

$$E$-$HA^\omega_{st} \vdash \Phi_B(a; b) \land a \leq^* c \rightarrow \Phi_B(c; b).$$
Characteristic principles

Definition

- $mAC^\omega \equiv \forall^{st} x \exists^{st} y \Phi \rightarrow \exists^{st} Y \forall^{st} x \exists^{st} y \leq^* Yx \Phi$;
- $R^\omega \equiv \forall x \exists^{st} y \Phi \rightarrow \exists^{st} z \forall x \exists y \leq^* z \Phi$;
- $I^\omega \equiv \exists^{st} z \exists x \forall y \leq^* z \phi \rightarrow \exists x \forall^{st} y \phi$;
- $IP_{\forall^{st}}^\omega \equiv (\exists^{st} x \phi \rightarrow \exists^{st} y \Psi) \rightarrow \exists^{st} z (\exists^{st} x \phi \rightarrow \exists^{st} y \leq^* z \Psi)$;
- $M^\omega \equiv (\exists^{st} x \phi \rightarrow \psi) \rightarrow \exists^{st} y (\exists x \leq^* y \phi \rightarrow \psi)$;
- $BUD^\omega \equiv \exists^{st} u, v (\forall x \leq^* u \phi \lor \forall y \leq^* v \psi) \rightarrow \forall^{st} x \phi \lor \forall^{st} y \psi$;
- $MAJ^\omega \equiv \forall^{st} x \exists^{st} y (x \leq^* y)$. 
Proposition

- $\text{E-HA}_{st}^\omega + I^\omega \vdash \text{BUD}^\omega$.
- $\text{E-HA}_{st}^\omega + R^\omega \vdash \text{MAJ}^\omega$. 
Soundness

Theorem (soundness theorem of $\mathcal{B}$)

For all formulas $\Phi$ of $\text{E-HA}_\text{st}^\omega$, if

$$\text{E-HA}_\text{st}^\omega + P \vdash \Phi,$$

then there are closed monotone terms $t$ of appropriate types such that

$$\text{E-HA}_\text{st}^\omega \vdash \forall^\text{st} b \Phi_B(t; b).$$

Abbreviation

$P := \text{mAC}^\omega + \text{R}^\omega + \text{I}^\omega + \text{IP}_{\forall^\text{st}}^\omega + \text{M}^\omega + \text{BUD}^\omega + \text{MAJ}^\omega.$
Characterization

Theorem (characterization theorem of B)

For all formulas $\Phi$ of $\text{E-HA}_{\text{st}}^\omega$, we have

$$\text{E-HA}_{\text{st}}^\omega + P \vdash \Phi \leftrightarrow \Phi^B.$$ 

Abbreviation

$P := \text{mAC}^\omega + R^\omega + I^\omega + \text{IP}_{\forall \text{st}}^\omega + M^\omega + \text{BUD}^\omega + \text{MAJ}^\omega.$
Transfer Principles

Definition

1. \((T\forall) \equiv \forall^{st} f (\forall^{st} x \phi \rightarrow \forall x \phi)\);
2. \((T\exists) \equiv \forall^{st} f (\exists x \phi \rightarrow \exists^{st} x \phi)\);

where \(f\) are all the free variables in the internal formula \(\phi\).
# Adding Transfer

<table>
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<th>Theorem</th>
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<tr>
<td><strong>1.</strong> Adding $T_{\forall}$ or $T_{\exists}$ to $E$-$HA_{\omega}^{\omega^*} + R + HGMP_{st}^{st}$ leads to nonconservativity over $HA$.</td>
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<tr>
<td><strong>2.</strong> Adding $T_{\forall}$ or $T_{\exists}$ to $E$-$HA_{st}^{\omega}$ leads to inconsistency.</td>
</tr>
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Krivine’s negative translation

$A^K \equiv \neg A_K \quad (\Phi_{at} \text{ is an atomic formula})$

$\forall (\Phi_{at})_K \equiv \neg \Phi_{at},$

$\forall (\neg \Phi)_K \equiv \neg \Phi_K,$

$\forall (\Phi \lor \Psi)_K \equiv \Phi_K \land \Psi_K,$

$\forall (\forall x \Phi)_K \equiv \exists x \Phi_K.$

Theorem (Soundness and characterization of $K$)

For all formulas $\Phi$ of the language of $E-\text{PA}_\omega^{st},$ we have:

1. $E-\text{PA}_\omega^{st} \vdash \Phi \Rightarrow E-\text{HA}_\omega^{st} + \text{I-LEM} \vdash \Phi^K;$
2. $E-\text{PA}_\omega^{st} \vdash \Phi \iff \Phi^K.$
Factorization

Theorem (factorisation $U = KB$)

For all formulas $\Phi$ of the language of $E-PA^\omega_{st}$, we have:

1. $E-HA^\omega_{st} + I-LEM \vdash \forall a, b \left( \Phi_U(a; b) \leftrightarrow \neg \forall c \leq^* b \left( \Phi_K)_B(a; c) \right) \right)$;
2. $E-HA^\omega_{st} + I-LEM \vdash \forall a, B \left( \Phi_U(a; Ba) \leftrightarrow (\Phi^K)_B(a; B) \right)$;
3. $E-HA^\omega_{st} + I-LEM + mAC^\omega_{st} \vdash \Phi^U \leftrightarrow (\Phi^K)_B$. 
Application

- Using the factorization $U = KB$ and the soundness theorem of $B$ one gets new proofs of the soundness and characterization theorems of $U$. 
Realizability with $q$-truth

Assigns to each formula $\Phi$ of $E$-$HA_{\omega}^{st}$ the formula $\Phi^{bq} : \equiv \exists^{st} a \Phi_{bq}(a)$ of $E$-$HA_{\omega}^{st}$ according to the following clauses, $\Phi^{bq} \equiv \exists^{st} a \Phi_{bq}(a)$ and $\Psi^{bq} \equiv \exists^{st} b \Psi_{bq}(b))$:

$$\phi^{bq} : \equiv [\phi],$$
$$st(t)^{bq} : \equiv \exists^{st} a [t \leq^* a],$$
$$(\Phi \land \Psi)^{bq} : \equiv \exists^{st} a, b [\Phi_{bq}(a) \land \Psi_{bq}(b)],$$
$$(\Phi \lor \Psi)^{bq} : \equiv \exists^{st} a, b [(\Phi_{bq}(a) \land \Phi) \lor (\Psi_{bq}(b) \land \Psi)],$$
$$(\Phi \rightarrow \Psi)^{bq} : \equiv \exists^{st} B \forall^{st} a [\Phi_{bq}(a) \land \Phi \rightarrow \Psi_{bq}(Ba)],$$
$$(\forall x \Phi)^{bq} : \equiv \exists^{st} a [\forall x \Phi_{bq}(a)],$$
$$(\exists x \Phi)^{bq} : \equiv \exists^{st} a [\exists x (\Phi_{bq}(a) \land \Phi)].$$
Realizability with $t$-truth

\[
\begin{align*}
\phi^{bt} & \equiv [\phi], \\
st(t)^{bt} & \equiv \tilde{\exists}^{st} a [t \leq^* a], \\
(\Phi \land \Psi)^{bt} & \equiv \tilde{\exists}^{st} a, b [\Phi^{bt}(a) \land \Psi^{bt}(b)], \\
(\Phi \lor \Psi)^{bt} & \equiv \tilde{\exists}^{st} a, b [\Phi^{bt}(a) \lor \Psi^{bt}(b)], \\
(\Phi \to \Psi)^{bt} & \equiv \tilde{\exists}^{st} B \tilde{\forall}^{st} a [(\Phi^{bt}(a) \to \Psi^{bt}(Ba)) \land (\Phi \to \Psi)], \\
(\forall x \Phi)^{bt} & \equiv \tilde{\exists}^{st} a [\forall x \Phi^{bt}(a)], \\
(\exists x \Phi)^{bt} & \equiv \tilde{\exists}^{st} a [\exists x \Phi^{bt}(a)].
\end{align*}
\]
Theorem

For all formulas $\Phi$ of $E\text{-}HA_{st}^{\omega}$, we have

$$E\text{-}HA_{st}^{\omega} \vdash \forall^st \ a(\Phi_{bt}(a) \leftrightarrow \Phi_{bq}(a) \land \Phi).$$
Soundness of \( bq \) and \( bt \)

**Theorem**

_For all formulas \( \Phi \) of \( E-HA_{st}^\omega \), if_

\[
E-HA_{st}^\omega \pm mAC^\omega \pm R^\omega \pm IP_{\not\exists}^\omega \pm MAJ^\omega \vdash \Phi,
\]

_\text{then there are closed monotone terms } t \text{ such that}_

\[
E-HA_{st}^\omega \pm mAC^\omega \pm R^\omega \pm IP_{\not\exists}^\omega \pm MAJ^\omega \vdash \Phi_{bq}(t),
\]

\[
E-HA_{st}^\omega \pm mAC^\omega \pm R^\omega \pm IP_{\not\exists}^\omega \pm MAJ^\omega \vdash \Phi_{bt}(t).
\]
Characterization of $bq$ and $bt$

**Theorem**

*For all formulas $\Phi$ of $E-HA^\omega_{st}$, we have*

$E-HA^\omega_{st} + mAC^\omega + R^\omega + IP^\omega_{\#st} + MAJ^\omega \vdash \Phi^{bq} \leftrightarrow \Phi,$

$E-HA^\omega_{st} + mAC^\omega + R^\omega + IP^\omega_{\#st} + MAJ^\omega \vdash \Phi^{bt} \leftrightarrow \Phi.$
Intuitionistic nonstandard bounded functional interpretation with \( q \)-truth

\[
\begin{align*}
\Phi^{Bq} & \equiv [\Phi], \\
st(t)^{Bq} & \equiv \exists^{st} a [t \leq^* a], \\
(\Phi \land \Psi)^{Bq} & \equiv \exists^{st} a, c \exists^{st} b, d [\Phi^{Bq}(a; b) \land \Psi^{Bq}(c; d)], \\
(\Phi \lor \Psi)^{Bq} & \equiv \exists^{st} a, c \exists^{st} e, f [\exists^{st} b \leq^* e \Phi^{Bq}(a; b) \land \Phi \lor (\exists^{st} d \leq^* f \Psi^{Bq}(c; d) \land \Psi)], \\
(\Phi \rightarrow \Psi)^{Bq} & \equiv \exists^{st} C, B \exists^{st} a, d [\exists^{st} b \leq^* Bad \Phi^{Bq}(a; b) \land \Phi \rightarrow \Psi^{Bq}(Ca; d)], \\
(\forall x \Phi)^{Bq} & \equiv \exists^{st} a \exists^{st} b [\forall x \Phi^{Bq}(a; b)], \\
(\exists x \Phi)^{Bq} & \equiv \exists^{st} a \exists^{st} c [\exists x (\exists^{st} b \leq^* c \Phi^{Bq}(a; b) \land \Phi)].
\end{align*}
\]
Intuitionistic nonstandard bounded functional interpretation with \(t\)-truth

\[
\Phi^{Bt} \equiv [\Phi],
\]
\[
st(t)^{Bt} \equiv \exists^{st} a \,[\, t \leq^* a \,],
\]
\[
(\Phi \land \Psi)^{Bt} \equiv \exists^{st} a, c \, \exists^{st} b, d \,[\, \Phi^{Bt}(a; b) \land \Psi^{Bt}(c; d) \,],
\]
\[
(\Phi \lor \Psi)^{Bt} \equiv \exists^{st} a, c \, \exists^{st} e, f \,[\, \exists b \leq^* e \, \Phi^{Bt}(a; b) \lor \exists d \leq^* f \, \Psi^{Bt}(c; d) \,],
\]
\[
(\Phi \rightarrow \Psi)^{Bt} \equiv \exists^{st} C, B \, \exists^{st} a, d \,[\, \exists b \leq^* Bad \, \Phi^{Bt}(a; b) \rightarrow \Psi^{Bt}(Ca; d) \land (\Phi \rightarrow \Psi) \,],
\]
\[
(\forall x \, \Phi)^{Bt} \equiv \exists^{st} a \, \exists^{st} b \,[\, \forall x \, \Phi^{Bt}(a; b) \,],
\]
\[
(\exists x \, \Phi)^{Bt} \equiv \exists^{st} a \, \exists^{st} c \,[\, \exists x \, \exists b \leq^* c \, \Phi^{Bt}(a; b) \,].
\]
Factorization

Theorem

For all formulas $\Phi$ of $E\text{-}HA^\omega_{st}$, we have

$$E\text{-}HA^\omega_{st} \vdash \forall_{st} a, b \left( \Phi_{Bt}(a; b) \leftrightarrow \Phi_{Bq}(a; b) \land \Phi \right).$$
Soundnesses of $B_q$ and $B_t$

**Theorem**

For all formulas $\Phi$ of $E\text{-}HA^\omega_{st}$, if

$$P \vdash \Phi,$$

then there are closed monotone terms $t$ such that

$$P \vdash \tilde{\forall}^{st} b \, \Phi_{B_q}(t; b),$$

$$P \vdash \tilde{\forall}^{st} b \, \Phi_{B_t}(t; b).$$

**Abbreviation**

$$P := E\text{-}HA^\omega_{st} \pm mAC^\omega \pm R^\omega \pm I^\omega \pm IP^{\forall\forall}_{st} \pm M^\omega \pm BUD^\omega \pm MAJ^\omega.$$
No optimal characterisation theorem of $B_q$ and $B_t$. 
No optimal characterisation theorem of $B_q$ and $B_t$.

*(optimal here means that it characterizes the least theory containing $E$-$\text{HA}^\omega_{st}$ and proving $\Phi^{B_q} \leftrightarrow \Phi$ for all formulas $\Phi$ of $E$-$\text{HA}^\omega_{st}$)*
No optimal characterisation theorem of $Bq$ and $Bt$.

No surprise! It is well-known that there are difficulties in proving optimal characterisation theorems for functional interpretations with truth.
Outline

Amuse-bouche

BFI

First course: functional interpretations for NSA
  Nonstandard analysis in proof theory
  Nonstandard Realizability
  Nonstandard Intuitionistic functional interpretation

Second course: a parametrised interpretation
  Parametrised interpretations of AL
  Parametrised interpretations of IL
  Instances

Dessert: realizability with stateful computations for NSA
Functional interpretations: applications

- Relative consistency of HA (Gödel)
- Independence of Markov’s principle (Kreisel)
- Proof mining (Kohlenbach)
- Interpretation of Weak König’s Lemma (Ferreira, Oliva)
- Interpretation of principles of Nonstandard analysis (Van den Berg, Briseid, Safarik)
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Different interpretations for different purposes.
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Different interpretations for different purposes.

We try to capture their common structure.
A pot-pourri of interpretations

- Kleene (numerical realizability) (1952)
- Gödel (Dialectica) (1958)
- Kreisel (modified realizability) (1959)
- Diller and Nahm (variant to avoid the contraction problem) (1974)
- Stein (family of interpretations) (1979)
- Ferreira and Oliva (bounded functional interpretation) (2005)
- Van den Berg, Briseid and Safarik (Herbrandized) (2012)
- ...
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- ...
Goal

Give a parametrised functional interpretation to unify all the well-known functional interpretations (including the approximate ones).
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▶ Compare the various existing functional interpretations.
▶ Help explain subtle details of the more recent interpretations (BFI, Herbrandized,...)
▶ Obtain new interpretations
Parametrised interpretations of $\mathcal{I}_s$ into $\mathcal{I}_t$
(jww P. Oliva)

$\mathcal{I}_s$ \xrightarrow{\{{{\cdot}}\}} \mathcal{I}_t$

$\mathcal{I}_s \cong \mathcal{I}_s$
$\mathcal{I}_t \cong \mathcal{I}_t$

$\mathcal{I}_s$: (intuitionistic) source theory
$\mathcal{I}_t$: (intuitionistic) target theory
$(\cdot)^\bullet; (\cdot)^\circ$: Girard’s translations
Parametrised interpretations of $\mathcal{I}_s$ into $\mathcal{I}_t$

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Parametrised interpretations of $\mathcal{I}_s$ into $\mathcal{I}_t$

$\mathcal{I}_s \xrightarrow{\{\cdot\}\_y; (\cdot)\_y^x} \mathcal{I}_t$

$\mathcal{I}_s^\bullet \sim \mathcal{I}_s^\circ \xrightarrow{\cdot \mid^x_y} \mathcal{I}_t^\bullet \sim \mathcal{I}_t^\circ$

$\mathcal{I}_s$: (intuitionistic) source theory
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$(\cdot)\bullet; (\cdot)^\circ$: Girard’s translations
Parametrised interpretations of $\mathcal{I}_s$ into $\mathcal{I}_t$

$\mathcal{I}_s \xrightarrow{{\{\cdot}\}^x; (\cdot)^x_y} \mathcal{I}_t$

$\mathcal{I}_s^\bullet \simeq \mathcal{I}_s^\circ$ \hspace{1cm} $\mathcal{I}_t^\bullet \simeq \mathcal{I}_t^\circ$

$\mathcal{I}_s^\bullet \xrightarrow{|\cdot|^x_y} \mathcal{I}_t^\bullet$

$\mathcal{I}_s$: (intuitionistic) source theory
$\mathcal{I}_t$: (intuitionistic) target theory
$(\cdot)^\bullet; (\cdot)^\circ$: Girard’s translations
Parametrised interpretations of $\mathcal{I}_s$ into $\mathcal{I}_t$:

\[ \mathcal{I}_s \xrightarrow{{\{\cdot\}\_x; (\cdot)\_y}} \mathcal{I}_t \]

\[ (\cdot)^\bullet; (\cdot)^\circ \]

\[ \mathcal{I}_s^\bullet \simeq \mathcal{I}_s^\circ \]

\[ \mathcal{I}_t^\bullet \simeq \mathcal{I}_t^\circ \]

$\mathcal{I}_s$: (intuitionistic) source theory

$\mathcal{I}_t$: (intuitionistic) target theory

$(\cdot)^\bullet; (\cdot)^\circ$: Girard’s translations
### AL Rules

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(id)</td>
<td>$A \vdash A$</td>
</tr>
<tr>
<td>(efq)</td>
<td>$\Gamma, \bot \vdash A$</td>
</tr>
<tr>
<td>(cut)</td>
<td>$\Gamma \vdash A \quad \Delta, A \vdash B \quad \Gamma, \Delta \vdash B$</td>
</tr>
<tr>
<td>(per)</td>
<td>$\Gamma \vdash A$ \quad $\pi{\Gamma} \vdash A$</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(⊗R)</td>
<td>$\Gamma \vdash A \quad \Delta \vdash B \quad \Gamma, \Delta \vdash A \otimes B$</td>
</tr>
<tr>
<td>(⊗L)</td>
<td>$\Gamma, A \otimes B \vdash C$</td>
</tr>
<tr>
<td>(→R)</td>
<td>$\Gamma, A \vdash B \quad \Gamma \vdash A \rightarrow B$</td>
</tr>
<tr>
<td>(→L)</td>
<td>$\Gamma \vdash A \quad \Delta, B \vdash C \quad \Gamma, \Delta, A \rightarrow B \vdash C$</td>
</tr>
</tbody>
</table>
**AL Rules**

\[
\frac{\Gamma \vdash A}{\Gamma \vdash \forall x A} \quad (\forall R, x \not\in \text{FV}(\Gamma))
\]

\[
\frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x A} \quad (\exists R)
\]

\[
\frac{\Gamma, A \vdash B}{\Gamma, \forall x A \vdash B} \quad (\forall L)
\]

\[
\frac{\Gamma, A \vdash B}{\Gamma, \exists x A \vdash B} \quad (\exists L, x \not\in \text{FV}(\Gamma, B))
\]

\[
\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B} \quad \text{(con)}
\]

\[
\frac{\Gamma \vdash B}{\Gamma, !A \vdash B} \quad \text{(wkn)}
\]

\[
\frac{!\Gamma \vdash A}{!\Gamma \vdash !A} \quad \text{(!R)}
\]

\[
\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B} \quad \text{(!L)}
\]
AL Rules

\[ \frac{\Gamma \vdash A}{\Gamma \vdash \forall x A} \quad (\forall R, x \not\in \text{FV}(\Gamma)) \]

\[ \frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x A} \quad (\exists R) \]

\[ \frac{\Gamma, A[t/x] \vdash B}{\Gamma \vdash A[t/x]} \quad (\forall L) \]

\[ \frac{\Gamma, A \vdash B}{\Gamma, \exists x A \vdash B} \quad (\exists L, x \not\in \text{FV}(\Gamma, B)) \]

\[ \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B} \quad \text{(con)} \]

\[ \frac{\Gamma \vdash B}{\Gamma, A \vdash B} \quad \text{(wkn)} \]

\[ \frac{!\Gamma \vdash A}{\Gamma \vdash !A} \quad \text{(!R)} \]

\[ \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B} \quad \text{(!L)} \]
<table>
<thead>
<tr>
<th>Rule</th>
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<tbody>
<tr>
<td>( \Gamma \vdash A )</td>
<td>(( \forall, x \not\in \text{FV}(\Gamma) ))</td>
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<tr>
<td>( \Gamma \vdash \forall x A )</td>
<td></td>
</tr>
<tr>
<td>( \Gamma \vdash A[t/x] )</td>
<td>(( \exists R ))</td>
</tr>
<tr>
<td>( \Gamma \vdash \exists x A )</td>
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</tr>
<tr>
<td>( \Gamma, A \vdash B )</td>
<td>(( \forall L ))</td>
</tr>
<tr>
<td>( \Gamma, \forall x A \vdash B )</td>
<td></td>
</tr>
<tr>
<td>( \Gamma, \exists x A \vdash B )</td>
<td></td>
</tr>
<tr>
<td>( \Gamma, !A, !A \vdash B )</td>
<td>(( \text{con} ))</td>
</tr>
<tr>
<td>( \Gamma, !A \vdash B )</td>
<td></td>
</tr>
<tr>
<td>( \Gamma \vdash B )</td>
<td>(( \text{wkn} ))</td>
</tr>
<tr>
<td>( !\Gamma \vdash A )</td>
<td>(( \text{!R} ))</td>
</tr>
<tr>
<td>( \Gamma, !A \vdash B )</td>
<td>(( \text{!L} ))</td>
</tr>
<tr>
<td>( \Gamma \vdash B )</td>
<td></td>
</tr>
<tr>
<td>( \Gamma, !A \vdash B )</td>
<td></td>
</tr>
</tbody>
</table>
We use Girard’s translations of $\textbf{IL}^B$ into $\textbf{AL}^B$:

\[
\begin{align*}
(P(x))^\bullet &\equiv P(x), \quad \text{if } P \not\equiv \bot \\
\bot^\bullet &\equiv \bot \\
(A \land B)^\bullet &\equiv A^\bullet \otimes B^\bullet \\
(A \to B)^\bullet &\equiv !A^\bullet \multimap B^\bullet \\
(\forall x A)^\bullet &\equiv \forall x A^\bullet \\
(\exists x A)^\bullet &\equiv \exists x !A^\bullet
\end{align*}
\]
From $\textbf{IL}^B$ into $\textbf{AL}^B$

We use Girard’s translations of $\textbf{IL}^B$ into $\textbf{AL}^B$:

\[
(P(x))^\bullet \equiv P(x) \quad (P(x))^\circ \equiv !P(x), \quad \text{if } P \neq \bot
\]

\[
\bot^\bullet \equiv \bot \quad \bot^\circ \equiv \bot
\]

\[
(A \land B)^\bullet \equiv A^\bullet \otimes B^\bullet \quad (A \land B)^\circ \equiv A^\circ \otimes B^\circ
\]

\[
(A \rightarrow B)^\bullet \equiv !A^\bullet \rightarrow B^\bullet \quad (A \rightarrow B)^\circ \equiv !(A^\circ \rightarrow B^\circ)
\]

\[
(\forall x A)^\bullet \equiv \forall x A^\bullet \quad (\forall x A)^\circ \equiv !\forall x A^\circ
\]

\[
(\exists x A)^\bullet \equiv \exists x! A^\bullet \quad (\exists x A)^\circ \equiv \exists x A^\circ
\]
From \( \text{IL}^B \) into \( \text{AL}^B \)

**Proposition**

If \( \Gamma \vdash \mathcal{I} A \) then \( !\Gamma \bullet \vdash \mathcal{I} \bullet A \bullet \) and \( \Gamma^\circ \vdash \mathcal{I}^\circ A^\circ \).
From $IL^B$ into $AL^B$

Proposition

If $\Gamma \vdash_I A$ then $!\Gamma^\circ \vdash_{I^\circ} A^\circ$ and $\Gamma^\circ \vdash_{I^\circ} A^\circ$.

Proposition (Gaspar, Oliva (2010))

$A^\circ$ is equivalent to $!A^\circ$ in $AL^B$. More precisely,

(i) $!A^\bullet \vdash_{AL^B} A^\circ$

(ii) $A^\circ \vdash_{AL^B} A^\bullet$
Define a translation of formulas of $\textbf{AL}^B$ into formulas of $\textbf{IL}^B$ inductively as follows:

$$
\begin{align*}
(P(x))^F & \equiv P(x), \text{ for the predicate symbols } P \\
(A \otimes B)^F & \equiv A^F \land B^F \\
(A \rightarrow B)^F & \equiv A^F \rightarrow B^F \\
(!A)^F & \equiv A^F \\
(\forall x A)^F & \equiv \forall x A^F \\
(\exists x A)^F & \equiv \exists x A^F
\end{align*}
$$
Towards the parametrised interpretation

Our parametrisd interpretation of $A_s$ into $A_t$ will contain three groups of parameters:

1. **Interpretation of computational predicate symbols**: For computational $P(x)$, associate, $x \prec^P a$.
Towards the parametrised interpretation

Our parametrised interpretation of $\mathcal{A}_s$ into $\mathcal{A}_t$ will contain three groups of parameters:

1. **Interpretation of computational predicate symbols**: For computational $P(x)$, associate, $x \prec^P a$.

2. **Domain of witnesses and counter-witnesses**: For each finite type $\tau$ we associate in $\mathcal{A}_t$ a formula $W_{\tau}(x)$, which we will use to restrict the domain of the witnesses and counter-witnesses. We assume combinatorial completeness for $W$
Towards the parametrised interpretation

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3. **Interpretation of $!A$**: A form of bounded quantification $\forall x \sqsubseteq_\tau a A$ satisfying:
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   $(Q_1)$ If $A \vdash_{A_t} B$ then $!\forall x \sqsubseteq_\tau a A \vdash_{A_t} \forall x \sqsubseteq_\tau a B$
Towards the parametrised interpretation

Our parametrised interpretation of $A_s$ into $A_t$ will contain three groups of parameters:

1. **Interpretation of computational predicate symbols**: For computational $P(x)$, associate $x \prec^P a$.

2. **Domain of witnesses and counter-witnesses**: For each finite type $\tau$ we associate in $A_t$ a formula $W_\tau(x)$, which we will use to restrict the domain of the witnesses and counter-witnesses.
   We assume combinatorial completeness for $W$.

3. **Interpretation of $!A$**: A form of bounded quantification $\forall x \sqcap_\tau a A$ satisfying:
   
   $$(Q_1) \quad \text{If } A \vdash_{A_t} B \text{ then } !\forall x \sqcap_\tau a A \vdash_{A_t} \forall x \sqcap_\tau a B$$
   
   $$(Q_2) \quad \vdash_{A_t} \forall x \sqcap_\tau a W(x)$$
Towards the parametrised interpretation

Finally, for each formula, terms $\eta(\cdot), (\cdot) \sqcup (\cdot)$ and $(\cdot) \circ (\cdot)$ satisfying conditions
Towards the parametrised interpretation

Finally, for each formula, terms \( \eta(\cdot), (\cdot) \sqcup (\cdot) \) and \( (\cdot) \circ (\cdot) \) satisfying conditions

\( (C_\eta) \leadsto \) to deal with substitutions.

\( (C_\sqcup) \leadsto \) to have a sort of union/maximum of two terms.

\( (C_\circ) \leadsto \) to deal with application of terms.
Parametrised $\mathbf{AL}$-interpretation

For each formula $A$ of $\mathcal{A}_s$, let us associate a formula $|A|^x_y$ of $\mathcal{A}_t$, with two fresh lists of free-variables $x$ and $y$, inductively as follows:

$$|P(x)|^a \equiv x \prec^P a, \quad (P \text{ computational})$$
$$|P(x)| \equiv P(x), \quad (P \text{ non-computational})$$

$$|A \rightarrow B|^{f,g}_{x,w} \equiv |A|^x_{g x w} \rightarrow |B|^f_w$$
$$|A \otimes B|^{x,v}_{y,w} \equiv |A|^x_{y} \otimes |B|^v_w$$
$$|\exists z A|^x_{y} \equiv \exists z |A|^x_y$$
$$|\forall z A|^x_{y} \equiv \forall z |A|^x_y$$
$$|! A|^x_a \equiv \forall y \sqsubseteq^\tau_A a |A|^x_y.$$
Witnessable AL sequents

A sequent $\Gamma \vdash A$ of $\mathcal{A}_s$ is said to be witnessable in $\mathcal{A}_t$ if there are closed terms $\gamma, a$ of $\mathcal{A}_t$ such that

(i) $\vdash_{\mathcal{A}_t} W(\gamma)$ and $\vdash_{\mathcal{A}_t} W(a)$

(ii) $!W(x, w), |\Gamma|_{xw}^x \vdash_{\mathcal{A}_t} |A|_{w}^{ax}$
Soundness

Theorem (Soundness)

If $\mathcal{A}_t$ is adequate and the axioms of $\mathcal{A}_s$ are witnessable in $\mathcal{A}_t$, then the parametrised $\textbf{AL}$-interpretation is sound.
**IL-interpretations**

Given an **AL**-interpretation $A \mapsto |A|^x_y$ based on the translated parameters we can derive two **IL**-interpretations, namely

$$A \mapsto (|A^\bullet|^x_y)^F \quad \text{and} \quad A \mapsto (|A^\circ|^x_y)^F$$

We will abbreviate these compound interpretations as

$$\{\{A\}\}_y^x \equiv (|A^\bullet|^x_y)^F \quad \text{and} \quad ((A))_y^x \equiv (|A^\circ|^x_y)^F$$
Proposition

\[
\begin{align*}
\{\{ P(x) \}\}^a & \equiv x \triangleleft^P a \quad \text{if } P \in \text{Pred}_A^c \\
\{\{ P(x) \}\} & \equiv P(x) \quad \text{if } P \in \text{Pred}_{A_s}^n \\
\{\{ A \rightarrow B \}\}_{x, w}^{f, g} & \equiv \forall y \sqsubseteq f x w \{\{ A\}\}^x_y \rightarrow \{\{ B\}\}^g_w^x \\
\{\{ A \land B \}\}_{y, w}^{x, v} & \equiv \{\{ A\}\}^x_y \land \{\{ B\}\}^v_w \\
\{\{ \exists z A \}\}_{y}^{x} & \equiv \exists z \forall y' \sqsubseteq y \{\{ A\}\}_{y'}^x \\
\{\{ \forall z A \}\}_{y}^{x} & \equiv \forall z \{\{ A\}\}_{y}^x
\end{align*}
\]
Parametrised interpretations of IL

Proposition

\[
\begin{align*}
\{\{P(x)\}\}^a & \equiv x \prec^P a \quad \text{if } P \in \text{Pred}^c_{A_s} \\
\{\{P(x)\}\} & \equiv P(x) \quad \text{if } P \in \text{Pred}^{nc}_{A_s} \\
\{\{A \rightarrow B\}\}^{f,g}_{x,w} & \equiv \forall y \sqsubseteq f x w \{\{A\}\}^x_y \rightarrow \{\{B\}\}^g_w \\
\{\{A \wedge B\}\}^{x,v}_{y,w} & \equiv \{\{A\}\}^x_y \wedge \{\{B\}\}^v_w \\
\{\{\exists z A\}\}^x_y & \equiv \exists z \forall y' \sqsubseteq y \{\{A\}\}^{x}_{y'} \\
\{\{\forall z A\}\}^x_y & \equiv \forall z \{\{A\}\}^x_y \\
\end{align*}
\]

In particular, we have that for computational predicate symbols $P$:

\[
\begin{align*}
\{\{\exists^P z A\}\}^{c,x}_{y} & \equiv \exists z \prec^P c \forall y' \sqsubseteq y \{\{A\}\}^{x}_{y'} \\
\{\{\forall^P z A\}\}^{f}_{b,y} & \equiv \forall z \prec^P b \{\{A\}\}^{f}_{y} \\
\end{align*}
\]
### Proposition

<table>
<thead>
<tr>
<th>Expression</th>
<th>Equivalent to</th>
</tr>
</thead>
<tbody>
<tr>
<td>(((P(x)))^a)</td>
<td>(x \prec^P a) if (P \in \text{Pred}^c_{\mathcal{A}_s})</td>
</tr>
<tr>
<td>(((P(x))))</td>
<td>(P(x)) if (P \in \text{Pred}^{nc}_{\mathcal{A}_s})</td>
</tr>
<tr>
<td>(((A \rightarrow B))_{x,w}^{f,g})</td>
<td>(\forall x', w' \sqcup x, w \ ((A)^{x'}<em>{f</em>{x'}w'} \rightarrow (B)^{g_{x'}}_{w'}))</td>
</tr>
<tr>
<td>(((A \land B))_{y,w}^{x,v})</td>
<td>(((A)^{x}<em>{y} \land (B)^{v}</em>{w})</td>
</tr>
<tr>
<td>(((\exists zA))_{x, w}^{y})</td>
<td>(\exists z((A)^{x}_{y})</td>
</tr>
<tr>
<td>(((\forall zA))_{y}^{x})</td>
<td>(\forall y' \sqcup y \ \forall z((A)^{x}_{y'})</td>
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</table>
### Parametrised interpretations of IL

<table>
<thead>
<tr>
<th>Proposition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$((P(x)))^a \iff x \prec^P a$ if $P \in \text{Pred}_{\mathcal{As}}^c$</td>
</tr>
<tr>
<td>$((P(x))) \iff P(x)$ if $P \in \text{Pred}_{\mathcal{As}}^{nc}$</td>
</tr>
<tr>
<td>$((A \to B))<em>{x,w}^{f,g} \iff \forall x', w' \sqsubset x, w ((A))</em>{x',w'}^{x'} \to (B))_{w'}^{g x'}$</td>
</tr>
<tr>
<td>$((A \land B))_{y,w}^{x,v} \iff (A)_y^x \land (B)_w^v$</td>
</tr>
<tr>
<td>$((\exists z A))_y^x \iff \exists z ((A))_y^x$</td>
</tr>
<tr>
<td>$((\forall z A))<em>y^x \iff \forall y' \sqsubset y \forall z ((A))</em>{y'}^x$</td>
</tr>
</tbody>
</table>

In particular, we have that for computational predicate symbols $P$:

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>$((\exists z^P A))_{y,c}^{x,c} \iff \exists z \prec^P c ((A))_y^x$</td>
</tr>
<tr>
<td>$((\forall z^P A))<em>{c,y}^{f,c} \iff \forall c', y' \sqsubset c, y \forall c'', y'' \sqsubset c', y' \forall z \prec^P c'' ((A))</em>{y''}^{f,c''}$</td>
</tr>
</tbody>
</table>
Comparing the interpretations

**Theorem**

For each formula $A$ there are tuples of closed terms $s_1$, $t_1$ and $s_2$, $t_2$ such that

(i) $W(x, y), \forall y' \sqsubseteq s_1 x y \{\{ A \}\}_y^x, \vdash_{\text{IL} \omega} ((A))^{t_1 x}_y$

(ii) $W(x, y), ((A))^{x}_{s_2 x y} \vdash_{\text{IL} \omega} \forall y' \sqsubseteq y \{\{ A \}\}^{t_2 x}_{y'}$

(iii) $\vdash_{\text{IL} \omega} W(s_1) \land W(s_2) \land W(t_1) \land W(t_2)$
### Instances

<table>
<thead>
<tr>
<th>( \forall x \sqsubseteq \tau ! a ! A )</th>
<th>( x &lt;^\tau a )</th>
<th>( W_\tau(a) )</th>
<th><strong>Interpretation</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>( A[a/x] )</td>
<td>( x = a )</td>
<td>true</td>
<td>Dialectica interpretation</td>
</tr>
<tr>
<td>( \forall x ! A )</td>
<td>( x = a )</td>
<td>true</td>
<td>Modified realizability</td>
</tr>
<tr>
<td>( \forall x \leq^* a ! A )</td>
<td>( x = a )</td>
<td>true</td>
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</tr>
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<td>( \forall x \in a ! A )</td>
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<td>Diller-Nahm interpretation</td>
</tr>
<tr>
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</tr>
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<td>Bounded modified realizability</td>
</tr>
<tr>
<td>( \forall x \leq^* a ! A )</td>
<td>( x \leq^* a )</td>
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</tr>
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</tr>
<tr>
<td>( A[a/x] )</td>
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<td>Herbrand Dialectica (( \simeq ) Dialectica)</td>
</tr>
<tr>
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<td>Herbrand realizability (for IL)</td>
</tr>
<tr>
<td>( \forall x \leq^* a ! A )</td>
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<td><strong>Herbrandized bfi</strong></td>
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Questions and future work

- Other ways to instantiate the parameters?
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- Characterization theorem?
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\[ \models A^x_a \equiv \forall y \in \tau a \models A^y \otimes A. \]
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\[ |!A|_a^x \equiv !\forall y \sqsubseteq \tau a |A|_y^x \otimes A. \]

- Interpretations for Nonstandard arithmetic: consider 2 types of atomic formulas.
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▸ Other ways to instantiate the parameters?
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▸ Interpretations for Nonstandard arithmetic: consider 2 types of atomic formulas.
▸ Composing with Krivine’s negative translation does one obtain classical interpretations? Factorization?
Outline

Amuse-bouche

BFI

First course: functional interpretations for NSA
   Nonstandard analysis in proof theory
   Nonstandard Realizability
   Nonstandard Intuitionistic functional interpretation

Second course: a parametrised interpretation
   Parametrised interpretations of AL
   Parametrised interpretations of IL
   Instances

Dessert: realizability with stateful computations for NSA
Goal: to deal with nonstandard analysis in the context of intuitionistic realizability, focusing on the Lightstone-Robinson construction of a model for nonstandard analysis through an ultrapower.

In particular, we consider an extension of the $\lambda$-calculus with a memory cell, that contains an integer (the state), in order to indicate in which slice of the ultrapower $M^\mathbb{N}$ the computation is being done.
Nonstandard models

1. Ground model
Nonstandard models

1. Ground model
2. Product of models

$$(u_n)_{n \in \mathbb{N}}$$
Nonstandard models

1. Ground model
2. Product of models
\((u_n)_{n \in \mathbb{N}}\)
Nonstandard models

1. Ground model
2. Product of models
$(u_n)_{n \in \mathbb{N}}$
Nonstandard models

1. Ground model
2. Product of models $(u_n)_{n \in \mathbb{N}}$
3. Quotient (w.r.t. $\mathcal{U}$) $[u_n]$
Nonstandard models

- Transfer
- Standardization
- Idealization
The first step in the Lightstone-Robinson construction aims at getting a product $\mathcal{M}^\mathbb{N}$ of the (initial) model $\mathcal{M}$.

▶ Add a memory cell to our calculus that contains an integer, which we call the *state*.

▶ The state keeps track of which “slice” of the product is the interpretation being done.

This product allows us to interpret first-order individuals as functions in $\mathbb{N}^\mathbb{N}$, so that the interpretation accounts for new elements – the so-called nonstandard elements – for instance the diagonal function.
Formulas \( A, B ::= \text{st}(e) \mid X(e_1, \ldots, e_n) \mid \text{Nat}(e) \mapsto A \mid A \to B \mid A \land B \mid A \lor B \mid \forall x. A \mid \exists x. A \mid \forall X. A \mid \exists X. A \)

Terms \( t, u ::= \ldots \mid \text{get} \mid \text{set} \)

States \( \mathcal{S} ::= \mathbb{N} \)

- get allows to read the current state
- set allows to increase the value of the current state
- With the exception of the get/set instructions, the syntax of terms does not account for states.
The interpretation of a formula $A$ together with a valuation $\rho$ is the set $|A|_\rho^\mathcal{S}$ defined inductively according to the following clauses:

\[ |\text{st}(e)|_\rho^\mathcal{S} \triangleq \begin{cases} \Lambda \times \mathcal{S} & \text{if } [e]_\rho \text{ is standard} \\ \emptyset & \text{otherwise} \end{cases} \]

\[ |X(e_1, \ldots, e_n)|_\rho^\mathcal{S} \triangleq \rho(X)@([e_1]_\rho, \ldots, [e_n]_\rho) \]

\[ |\{\text{Nat}(e)\} \mapsto A|_\rho^\mathcal{S} \triangleq \{(t; s) \in \Lambda \times \mathcal{S} : (t \vec{n}; s) \in |A|_\rho^\mathcal{S}, \text{ where } n = [e]_\rho(s)\} \]

\[ |A \rightarrow B|_\rho^\mathcal{S} \triangleq \{(t; s) \in \Lambda \times \mathcal{S} : \forall u.((u; s) \in |A|_\rho^\mathcal{S} \Rightarrow (t u; s) \in |B|_\rho^\mathcal{S})\} \]

\[ |A_1 \land A_2|_\rho^\mathcal{S} \triangleq \{(t; s) \in \Lambda \times \mathcal{S} : (\pi_1(t); s) \in |A_1|_\rho^\mathcal{S} \land (\pi_2(t); s) \in |A_2|_\rho^\mathcal{S}\} \]

\[ |A_1 \lor A_2|_\rho^\mathcal{S} \triangleq \{(t; s) \in \Lambda \times \mathcal{S} : \exists i \in \{1, 2\}.(\text{case } t \{t_1(x_1) \mapsto x_1 | t_2(x_2) \mapsto x_2\}; s) \in |A_i|_\rho^\mathcal{S}\} \]

\[ |\forall x.A|_\rho^\mathcal{S} \triangleq \bigcup_{f \in \mathcal{N}^\mathcal{S}} |A|_{\rho, x \mapsto f}^\mathcal{S} \quad |\forall X.A|_\rho^\mathcal{S} \triangleq \bigcap_{F: \mathbb{N}^k \rightarrow \mathsf{SAT}} |A|_{\rho, X \mapsto F}^\mathcal{S} \]

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This interpretation realizes (in a non-trivial way):

- Usual properties of nonstandard natural numbers (including external induction)
- The diagonal as a nonstandard element
- Idealization
- Transfer
- Overspill and Underspill

It does not validate Standardization: for that a quotient is necessary (work in progress).
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- What applications are there for the interpretations with truth? Can they give additional information about Transfer?
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- Is it possible/interesting to extend nonstandard interpretations to the feasible context?
- Adapt the interpretation with slices to Krivine's classical realizability (in progress)
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*(Notes not intended for publication)*

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Thank you!