The proof theory of substructural logics with fixpoints

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based on joint work with

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1. Fixpoint logic(s)
2. Linear logic with fixpoints
3. Complexity
4. Phase semantics

Fixpoint logic(s)

Different logic, different reasons

- Extensions of propositional modal logics: LTL, μ-calculus, ...

  to express richer specifications: "something happens infinitely often", "something happens after some time" and so on
- Extensions of first-order logic: FO[LFP], FO[IFP], ...

  to define richer classes of finite models and their descriptive complexity
- Extensions of categorical grammar: Kleene Algebra, Action algebra, ...

  to algebraically define various classes of formal languages

This talk: the proof theory of fixpoint logic(s)

Explicit (co)induction

We start by adding the μ and ν operators for lfp and gfp respectively such that $\mu x.\varphi = \neg \nu x.\neg\varphi$.

$$\frac{\varphi[\psi/x] \vdash \psi}{\mu x.\varphi \vdash \psi}\,(\mu_\ell) \qquad \frac{\Gamma \vdash \varphi[\mu x.\varphi/x], \Delta}{\Gamma \vdash \mu x.\varphi, \Delta}\,(\mu_r)$$

$$\frac{\Gamma, \varphi[\nu x.\varphi/x] \vdash \Delta}{\Gamma, \nu x.\varphi \vdash \Delta}\,(\nu_\ell) \qquad \frac{\psi \vdash \varphi[\psi/x]}{\psi \vdash \nu x.\varphi}\,(\nu_r)$$

We start by adding the μ and ν operators for lfp and gfp respectively such that $\mu x.\varphi = \neg\nu x.\neg\varphi$.

$$\frac{\varphi[\psi/x] \leq \psi}{\mu x.\varphi \leq \psi}(\mu_\ell) \qquad \frac{\Gamma \leq \varphi[\mu x.\varphi/x]}{\Gamma \leq \mu x.\varphi}(\mu_r)$$

$$\frac{\varphi[\nu x.\varphi/x] \leq \Delta}{\nu x.\varphi \leq \Delta}(\nu_\ell) \qquad \frac{\psi \leq \varphi[\psi/x]}{\psi \leq \nu x.\varphi}(\nu_r)$$

- μ_ℓ expresses that $\mu x.\varphi$ is smaller than any post fixpoint of φ. Dually ν_r expresses that $\nu x.\varphi$ is larger than any pre fixpoint of φ.
- μ_r expresses that $\mu x.\varphi$ is indeed a post fixpoint of φ and dually ν_ℓ expresses that $\nu x.\varphi$ is indeed a pre fixpoint of φ.

- Cut inadmissible in system with explicit (co)induction without weakening. The proof below has no cut-free version.

$$\dfrac{\dfrac{\dfrac{}{a \vdash a}\,(\mathrm{id}) \quad \dfrac{}{a \vdash a}\,(\mathrm{id})}{a, a \vdash a \otimes a}\,(\otimes_r) \quad \dfrac{\dfrac{}{a \otimes a \vdash a \otimes a}\,(\mathrm{id})}{a \otimes a \vdash \nu x.x}\,(\nu_r)}{a, a \vdash \nu x.x}\,(\mathrm{cut})$$

- New rules:

$$\dfrac{\Gamma, \psi \vdash \Delta \quad \varphi[\psi/x] \vdash \psi}{\Gamma, \mu x.\varphi \vdash \Delta}\,(\mu_\ell^{ind}) \qquad \dfrac{\Gamma \vdash \Delta, \psi \quad \psi \vdash \varphi[\psi/x]}{\Gamma \vdash \Delta, \nu x.\varphi}\,(\nu_r^{ind})$$

- Choosing an appropriate ψ is akin choosing an appropriate (co)induction hypothesis.
- Cut admissibility does not guarantee subformulæproperty.

Implicit (co)induction

$$\frac{\Gamma, \varphi[\mu x.\varphi/x] \vdash \Delta}{\Gamma, \mu x.\varphi \vdash \Delta}\,(\mu_l) \qquad \frac{\Gamma \vdash \varphi[\mu x.\varphi/x], \Delta}{\Gamma \vdash \mu x.\varphi, \Delta}\,(\mu_r)$$

$$\frac{\Gamma, \varphi[\nu x.\varphi/x] \vdash \Delta}{\Gamma, \nu x.\varphi \vdash \Delta}\,(\nu_l) \qquad \frac{\Gamma \vdash \varphi[\nu x.\varphi/x], \Delta}{\Gamma \vdash \nu x.\varphi, \Delta}\,(\nu_r)$$

Implicit (co)induction

$$\frac{\Gamma, \varphi[\mu x.\varphi/x] \leq \Delta}{\Gamma, \mu x.\varphi \leq \Delta}(\mu_l) \qquad \frac{\Gamma \leq \varphi[\mu x.\varphi/x], \Delta}{\Gamma \leq \mu x.\varphi, \Delta}(\mu_r)$$

$$\frac{\Gamma, \varphi[\nu x.\varphi/x] \leq \Delta}{\Gamma, \nu x.\varphi \leq \Delta}(\nu_l) \qquad \frac{\Gamma \leq \varphi[\nu x.\varphi/x], \Delta}{\Gamma \leq \nu x.\varphi, \Delta}(\nu_r)$$

- μ_ℓ and μ_r expresses that $\mu x.\varphi$ is a pre fixpoint and post fixpoint of φ respectively.
- Similarly for ν_ℓ and ν_r.

Hang on!

- $\mu x.\varphi$ and $\nu x.\varphi$ are indeed fixpoints but not necessarily least and greatest.
- $\nu x.x$ cannot be proven.

- Let's allow proof trees of infinite height.
- Now $\nu x.x$ can be proved:

$$\dfrac{\dfrac{\vdots}{\vdash \nu x.x}\,(\nu)}{\vdash \nu x.x}\,(\nu)$$

- But inconsistent!

$$\dfrac{\dfrac{\dfrac{\vdots}{\mu x.x}\,(\mu)}{\mu x.x}\,(\mu) \qquad \dfrac{\dfrac{\vdots}{\nu x.x, \Gamma}\,(\nu)}{\nu x.x, \Gamma}\,(\nu)}{\Gamma}\,(\text{cut})$$

Progress condition: Along every branch, the smallest formula occurring infinitely often is a ν-formula.

Circular proofs := Non-wellfounded proofs that have finitely distinct subtrees.

In terms of mathematical content, like a proof by infinite descent.

Brotherston-Simpson hypothesis

Induction is as powerful as infinite descent.

Regularisation hypothesis

Circular proofs are as powerful as non-wellfounded proofs.

Note that circular proofs arise in not just logics with fixed points. Notably, arithmetic, provability logics like Gödel-Lob logic, etc.

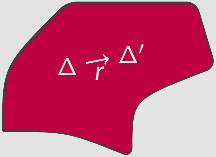
μLK REGULARISATION

Theorem

Non-wellfounded proofs and circular proofs prove exactly the same set of μLK theorems.

Game $\mathcal{G}_\Gamma$

- Arena is the set of all possible sequents in a proof of Γ.
- Prover chooses an inference rule r with conclusion the current state Δ.
- Denier chooses one of the premisses Δ'.
- A play is winning iff it starts from Γ and satisfies the progress condition.



Lemma

$\mathcal{G}_\Gamma$ is a parity game.

Proof idea

- Arena is finite (seqeunts are sets and made of subformulæ of Γ).
- The progress condition is a parity condition.

Proof of regularisation

There is a non-wellfounded proof of Γ $\Rightarrow$ Prover has a winning strategy in $\mathcal{G}_\Gamma$ $\Rightarrow$ Prover has a memoryless winning strategy (Determinacy of parity games) $\Rightarrow$ There is a circular proof of Γ. □

Let $\varphi = \nu x.x \parr x$

$$
\cfrac{\cfrac{\cfrac{\cfrac{\vdots}{\vdash \varphi, \varphi, \varphi}}{\vdash \varphi \parr \varphi, \varphi}\;(\parr)}{\vdash \varphi, \varphi}\;(\nu)}{\cfrac{\vdash \varphi \parr \varphi}{\vdash \varphi}\;(\nu)}\;(\parr)
\qquad
\cfrac{\cfrac{\cfrac{\cfrac{\vdash \varphi, \mu x.x \qquad \cfrac{\vdash \nu x.x, \varphi}{\vdash \nu x.x, \varphi}\;(\nu)}{\vdash \varphi, \varphi, \mu x.x}\;(\mathrm{cut})}{\vdash \varphi \parr \varphi, \mu x.x}\;(\parr)}{\vdash \varphi, \mu x.x}\;(\nu) \qquad \cfrac{\vdash \nu x.x}{\vdash \nu x.x}\;(\nu)}{\vdash \varphi}\;(\mathrm{cut})
$$

- The first proof is non-wellfounded since weakening is not allowed.
- In fact, there is no circular proof if cuts are not allowed!
- However, there is a circular proof with cuts.

Regularisation hypothesis

Circular proofs with cuts are as powerful as non-wellfounded proofs.

Linear logic with fixpoints

$$\frac{\vdash \Delta, \varphi, \varphi', \Delta'}{\vdash \Delta, \varphi', \varphi, \Delta'}(\mathrm{ex}) \quad \frac{\vdash \Delta, \varphi, \varphi}{\vdash \Delta, \varphi}(\mathrm{c}) \quad \frac{\vdash \Delta}{\vdash \Delta, \varphi}(\mathrm{w})$$

- **Exchange**: sequents as lists $\to$ sequents as multisets
- **Contraction**: sequent as multisets $\to$ sequent as sets

Substructural logic(s) := Logics where one or more of the structural rules are absent or only allowed under controlled circumstances.

- Establishes a direct connection between logic and type systems for models of computation.
- Can be seen at three levels:
  1. formulas $\leftrightarrow$ types.
  2. proof objects $\leftrightarrow$ programs.
  3. normalisation $\leftrightarrow$ computation/reduction.

Substructural type systems:= Type systems analogous to substructural logics

Exchange	Contraction	Weakening	Every variable is used
$\times$	$\times$	$\times$	Exactly once in the order introduced
$\checkmark$	$\times$	$\times$	Exactly once
$\checkmark$	$\times$	$\checkmark$	At most once
$\checkmark$	$\checkmark$	$\times$	At least once

Linear logic (MALL)

	conjunction	disjunction	"true"	"false"
multiplicative	$\otimes$	$\parr$	$\mathbf{1}$	$\bot$
additive	$\&$	$\oplus$	$\top$	$\mathbf{0}$

$$\frac{}{\vdash \varphi, \varphi^\perp}\,(\mathrm{id}) \qquad \frac{\vdash \Gamma_1, \varphi \quad \vdash \Gamma_2, \varphi^\perp}{\vdash \Gamma_1, \Gamma_2}\,(\mathrm{cut})$$

$$\frac{\vdash \Gamma, \varphi_1, \varphi_2}{\vdash \Gamma, \varphi_1 \parr \varphi_2}\,(\parr) \qquad \frac{\vdash \Gamma_1, \varphi_1 \quad \vdash \Gamma_2, \varphi_2}{\vdash \Gamma_1, \Gamma_2, \varphi_1 \otimes \varphi_2}\,(\otimes) \qquad \frac{\vdash \Gamma, \varphi_i}{\vdash \Gamma, \varphi_1 \oplus \varphi_2}\,(\oplus_i) \qquad \frac{\vdash \Gamma, \varphi_1 \quad \vdash \Gamma, \varphi_2}{\vdash \Gamma, \varphi_1 \& \varphi_2}\,(\&)$$

$$\frac{}{\vdash \mathbf{1}}\,(\mathbf{1}) \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\,(\bot) \qquad \frac{}{\vdash \Gamma, \top}\,(\top)$$

No rule for **0**

$$\mu\text{MALL} = \text{MALL} + \text{fixpoints}$$

Wellfounded system := $\mu\text{MALL}^{\text{ind}}$ Circular system := $\mu\text{MALL}^{\circlearrowright}$
Non-wellfounded system := μMALL^{∞}

Computation content of μMALL$^{\text{ind}}$

- $\mathbb{N} := \mu x.\mathbf{1} \oplus x$
- $List_{\mathbb{A}} := \mu x.\bot \oplus (\mathbb{A} \otimes x)$
- $Stream_{\mathbb{A}} := \nu x.\mathbb{A} \otimes x$

$$\mathbf{1} \equiv \dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\,(\mathbf{1})}{\vdash \mathbf{1} \oplus \mathbb{N}}\,(\oplus_1)}{\vdash \mathbb{N}}\,(\mu) \quad , \quad \mathbf{2} \equiv \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathbf{1}}\,(\mathbf{1})}{\vdash \mathbf{1} \oplus \mathbb{N}}\,(\oplus_1)}{\vdash \mathbb{N}}\,(\mu)}{\vdash \mathbf{1} \oplus \mathbb{N}}\,(\oplus_2)}{\vdash \mathbb{N}}\,(\mu) \quad , \ldots$$

Contraction's back on the menu, boys!

- In "full" linear logic there are *exponential* modalities that allow weakening and contractions.
- Exponentials can be encoded in μMALL (with a few caveats).
- Also, natural numbers encoded using fixpoints can be contracted:

$$\dfrac{\dfrac{}{\mathbb{N}\otimes\mathbb{N}\vdash\mathbb{N}\otimes\mathbb{N}}\,(\mathrm{id}) \quad \dfrac{\dfrac{}{\mathbb{N}\otimes\mathbb{N}\vdash\mathbb{N}\otimes\mathbb{N}}\,(\mathrm{id})}{\mathbf{1}\oplus(\mathbb{N}\otimes\mathbb{N})\vdash\mathbb{N}\otimes\mathbb{N}}\,(\oplus^2_\ell)}{\mathbb{N}\vdash\mathbb{N}\otimes\mathbb{N}}\,(\mu_\ell)$$

$\boxed{\text{Remember } \mathbb{N} := \mu x.\mathbf{1}\oplus x}$

- $\mathbb{N} := \mu x.\mathbf{1} \oplus x$
- $List_{\mathbb{A}} := \mu x.\bot \oplus (\mathbb{A} \otimes x)$
- $Stream_{\mathbb{A}} := \nu x.\mathbb{A} \otimes x$

$$\mathbf{1} :: \mathbf{2} :: \mathbf{3} \cdots \equiv \dfrac{\dfrac{\overset{\overset{\mathbf{1}}{\nabla}}{\vdash \mathbb{N}} \quad \dfrac{\dfrac{\overset{\overset{\mathbf{2}}{\nabla}}{\vdash \mathbb{N}} \quad \overset{\vdots}{\vdash Stream_{\mathbb{N}}}}{\vdash \mathbb{N} \otimes Stream_{\mathbb{N}}}(\otimes)}{\vdash Stream_{\mathbb{N}}}(\nu)}{\vdash \mathbb{N} \otimes Stream_{\mathbb{N}}}(\otimes)}{\vdash Stream_{\mathbb{N}}}(\nu) \qquad \mathbf{1}^{\omega} \equiv \dfrac{\dfrac{\overset{\overset{\mathbf{1}}{\nabla}}{\vdash \mathbb{N}} \quad \vdash Stream_{\mathbb{N}}}{\vdash \mathbb{N} \otimes Stream_{\mathbb{N}}}(\otimes)}{\vdash Stream_{\mathbb{N}}}(\nu)$$

- The first stream cannot be represented by a circular proof.

- Cut elimination $\leftrightarrow$ computation/evaluation in CH
- Stronger than *cut admissibility* which requires only the *existence* of a cut-free proof.
- Need to prove *productivity* instead of *termination*.

$$\dfrac{\begin{array}{c} n \\ \bigtriangledown \\ \vdash \mathbb{N} \end{array} \quad \begin{array}{c} f \\ \bigtriangledown \\ \mathbb{N} \vdash \mathit{Stream}_{\mathbb{N}} \end{array}}{\vdash \mathit{Stream}_{\mathbb{N}}}\,(\mathrm{cut}) \qquad \rightsquigarrow^{\infty} \qquad \begin{array}{c} n^{\omega} \\ \bigtriangledown \\ \vdash \mathit{Stream}_{\mathbb{N}} \end{array}$$

where $f :=$

$$\dfrac{\begin{array}{c} \mathit{dup} \\ \bigtriangledown \\ \mathbb{N} \vdash \mathbb{N} \otimes \mathbb{N} \end{array} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\mathbb{N} \vdash \mathbb{N}}\,(\mathrm{id}) \quad \mathbb{N} \vdash \mathit{Stream}_{\mathbb{N}}}{\mathbb{N}, \mathbb{N} \vdash \mathbb{N} \otimes \mathit{Stream}_{\mathbb{N}}}\,(\otimes_r)}{\mathbb{N}, \mathbb{N} \vdash \mathit{Stream}_{\mathbb{N}}}\,(\nu_r)}{\mathbb{N} \otimes \mathbb{N} \vdash \mathit{Stream}_{\mathbb{N}}}\,(\otimes_\ell)}{\mathbb{N} \vdash \mathit{Stream}_{\mathbb{N}}}\,(\mathrm{cut})$$

The landscape of μMALL

Abstract syntax

Ludics (BDS'15)
Game semantics (Clairambault'09)

Understanding cut elimination

Proof-nets (DS'19, DPS'21)
Bouncing threads (BDKS)

Denotational semantics

Coherence space semantics (EJ'21)
Categorical semantics (FS'13)

Provability

Complexity (DDS)
Truth Semantics (DJS)
Annotated sequents (NST'18)

Curry-Howard correspondence

Session types (DP)
Relation with System T (KPP'21)

Complexity

Theorem

$\mu\mathrm{MALL}^{\circlearrowright} \subsetneq \mu\mathrm{MALL}^{\infty}$

Proof idea

- Show that $\mu\mathrm{MALL}^{\infty}$ is Π^0_1-hard (consequently undecidable).
- $\mu\mathrm{MALL}^{\circlearrowright}$ is in Σ^0_1.
  (circular proofs are finitely representable, hence enumerable)
- If $\mu\mathrm{MALL}^{\infty} = \mu\mathrm{MALL}^{\circlearrowright}$, then $\Pi^0_1 \subseteq \Sigma^0_1$. Contradiction!

Non-constructive!

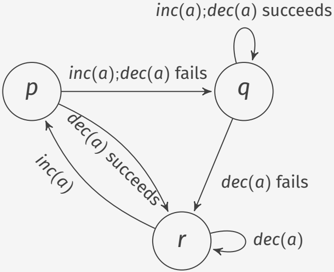
We do not exhibit an actual sequent

Fragment of linear logic	Complexity of Provability
MLL	NP-complete [Kanovich'91]
MALL	PSPACE-complete [LMSS'90]
MELL	?
LL	Undecidable [LMSS'90]

- In comparison, LK is NP-complete and LJ is PSPACE-complete.
- Exponentials can be encoded in μMALL. So, we expect it to be at least as hard as LL.

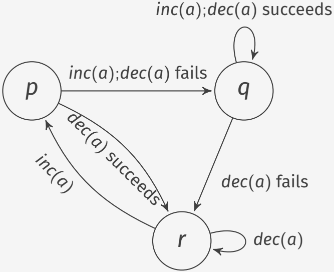
Counter machines

- Counter $a \in \mathbb{N}_0$ and $dec(a)$ fails if $a = 0$



- Halting of one counter automata decidable (Folklore)
- Halting of two counter automata Σ^0_1-complete (Minsky)

Reduction to Minsky machine



$$\boxed{[\![inc(a)]\!] := p^{\perp} \otimes (a \parr q)}$$

$$\dfrac{\vdash p, p^{\perp} \qquad \dfrac{\vdash a, a, a, q}{\vdash a, a, a \parr q}(\parr)}{\vdash a, a, p, p^{\perp} \otimes (a \parr q)}(\otimes)$$

Encode dec and zero-check.

$$\boxed{\varphi := \nu x. \bot \& (\bigoplus_{I \in \mathcal{M}} [\![I]\!] \parr x)}$$

Theorem

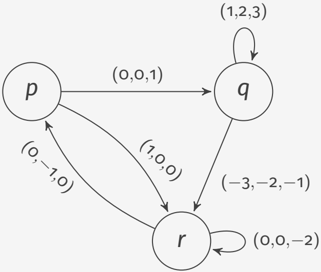
$\vdash p, \varphi$ provable iff $\mathcal{M}$ is non-halting.

Proof idea

($\Leftarrow$) This relies on being able to use $[\![I]\!]$ for every $I \in \mathcal{M}$.

($\Rightarrow$) This relies on cut admissibility and focussing (the ability to apply certain rules context-freely).

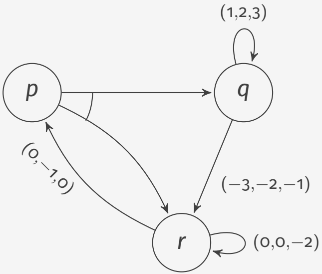
Vector addition systems (& extensions)



Starting from $\langle p, (0,0,0)\rangle$ is $\langle q, (10,10,10)\rangle$ reachable?

Theorem

Reachability of vector addition system with states is decidable and reduces from the provability of the Horn-fragment of MELL.



Branching VASS

- Multiplicative splitting

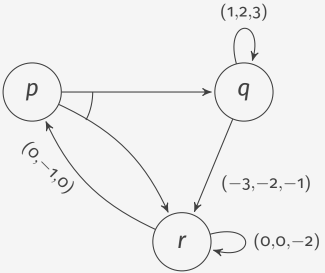
$$\frac{\langle q,(3,2,0)\rangle \quad \langle r,(1,0,0)\rangle}{\langle p,(4,2,0)\rangle}$$

- Set of accepting configurations called axioms

Starting from $\langle q,(10,10,10)\rangle$ is there a run tree ending in axioms?

Theorem

BVASS reachability is open and equivalent to the provability of MELL.



Alternating VASS

- Additive splitting

$$\frac{\langle q,(4,2,0)\rangle \quad \langle r,(4,2,0)\rangle}{\langle p,(4,2,0)\rangle}$$

- Set of accepting configurations called axioms

Starting from $\langle q,(10,10,10)\rangle$ is there a run tree ending in axioms?

Theorem

AVASS reachability is undecidable and reduces to μMALL provability with only μ.

What about Brotherston-Simpson?

Towards $\mu\text{MALL}^{\text{ind}} = \mu\text{MALL}^{\circlearrowleft}$

NTS'18[1] gives an annotated circular system that can be finitised.

Towards $\mu\text{MALL}^{\text{ind}} \neq \mu\text{MALL}^{\circlearrowleft}$

- Both $\mu\text{MALL}^{\text{ind}}$ and $\mu\text{MALL}^{\circlearrowleft}$ have the same complexity! Cannot use the complexity argument of non-regularisation.
- Suppose an oracle gives us a sequent that has a $\mu\text{MALL}^{\circlearrowleft}$ proof but not a $\mu\text{MALL}^{\text{ind}}$ proof. How do we verify this?
- Since provability is undecidable, there is no general algorithm!

[1] Rémi Nollet, Alexis Saurin, and Christine Tasson. "Local Validity for Circular Proofs in Linear Logic with Fixed Points". In: *27th EACSL Annual Conference on Computer Science Logic, CSL 2018, Birmingham, UK*. ed. by Dan R. Ghica and Achim Jung. Vol. 119. LIPIcs.

Phase semantics

- Establishes a semantic meaning of truth.
- Gives a mapping $[\![\bullet]\!]$: Formulas $\to$ Mathematical Object such that a formula is provable iff its interpretation satisfies some property.
- Via CH, corresponds to *type inhabitation*.

Example

- Truth semantics of LK : Boolean algebras
- Truth semantics of LJ : Heyting algebras
- Truth semantics of S4 : Boolean algebras with an interior operator

Truth semantics is basically the *Lindenbaum algebra* i.e. the quotient of logical formulas under provability equivalence.

Context matters!

$T \vdash \varphi \multimap \psi$ and $T \vdash \psi \multimap \chi$ does not imply $T \vdash \varphi \multimap \chi$ rather $T, T \vdash \varphi \multimap \chi$.

Define $\boxed{Pr(\varphi) := \{\Gamma \mid \vdash \Gamma, \varphi \text{ is provable}\}}$

- $Pr(\bot) =$ Set of all provable sequents.
- $Pr(\varphi \otimes \psi) = \{\Gamma_1 \uplus \Gamma_2 \mid \Gamma_1 \in Pr(\varphi), \Gamma_2 \in Pr(\psi)\} = Pr(\varphi).Pr(\psi)$
- $Pr(\varphi \& \psi) = Pr(\varphi) \cap Pr(\psi) \ldots$

The algebraic object we are after must be a monoid and a lattice (a.k.a *residuated semilattice*).

Phase space

A phase space is a commutative monoid M along with a $\perp\!\!\!\perp \subseteq M$.
Let $X, Y \subseteq M$. Define

$$XY := \{xy \mid x \in X, y \in Y\} \qquad X^{\perp} := \{z \mid \forall x \in X. xz \in \perp\!\!\!\perp\}$$

X is called a fact if $X^{\perp\perp} = X$.

We interpret formulas (and sequents) on facts.

$$[\![\varphi \otimes \psi]\!] = ([\![\varphi]\!].[\![\psi]\!])^{\perp\perp} \quad [\![\varphi ⅋ \psi]\!] = ([\![\varphi]\!]^{\perp}[\![\psi]\!]^{\perp})^{\perp}$$
$$[\![\varphi \& \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!] \qquad [\![\varphi \oplus \psi]\!] = ([\![\varphi]\!] \cup [\![\psi]\!])^{\perp\perp}$$

Theorem (Girard'87)

Γ *is provable in* MALL *iff* $1 \in [\![\Gamma]\!]$

- Let $M =$ Set of all sequents.
- Let $\Gamma, \Delta \in M$. Then, $\Gamma \cdot \Delta = \Gamma, \Delta$.
- Therefore, $(M, \cdot, \varnothing)$ is a monoid.
- Let $\perp\!\!\!\perp = Pr(\perp)$ and we have a phase space.

Lemma (Adequation lemma)

$[\![\Gamma]\!] \subseteq Pr(\Gamma)$

Completeness proof

$\varnothing \in [\![\Gamma]\!] \Rightarrow \varnothing \in Pr(\Gamma) \Rightarrow \vdash \Gamma, \varnothing$ is provable. $\square$

Fact

The set of facts is a complete lattice.

$\therefore$ We can interpret fixpoint formulas as:

$$[\![\mu x.\varphi]\!] = \mathit{lfp}(\lambda X.\varphi(X)) \quad [\![\nu x.\varphi]\!] = \mathit{gfp}(\lambda X.\varphi(X))$$

The interpretations are facts by Knaster-Tarski theorem.

Too liberal!

Not every fact is an image of $[\![\bullet]\!]$. So, $[\![\varphi(X)]\!]$ doesn't necessarily correspond to the interpretation of any formula.

Sound but not complete!

Restrict to a subset of fact closed under μMALL operations.

Theorem

Γ is provable in μMALL$^{\text{ind}}$ iff $1 \in [\![\Gamma]\!]$

Proof idea

($\Rightarrow$) Soundness is easy induction on the proof.

($\Leftarrow$) For completeness, we start from the syntactic monoid but induction on formulas does not work (due to absence of subformula property)! We use Girard's *candidates of reducibility*.

Define $\boxed{\langle\varphi\rangle := \{F \in \mathit{Facts} \mid F \subseteq \mathit{Pr}(\varphi)\}}$

Lemma (Adequation Lemma)

For all facts $F \in \langle\psi\rangle$, $[\![\varphi(F)]\!] \subseteq \mathit{Pr}(\varphi(\psi))$.

Closure ordinals

Let f be a monotonic function on a complete lattice $(\mathcal{L}, \leq, \top, \bot, \wedge, \vee)$. By Knaster-Tarski's theorem, it has a fixed point. But can we compute it?

$$\begin{aligned} \Theta_0 &= f(\bot); \\ \Theta_{\alpha+1} &= f(\Theta_\alpha); \\ \Theta_\lambda &= \bigwedge_{\alpha \in \lambda} \Theta_\alpha. \end{aligned}$$

The seqeuence $\Theta_0, \Theta_1, \ldots$ is ultimately stationary and the lfp of f. The smallest ordinal α such that $\Theta_\alpha = \Theta_{\alpha+1}$ is the **closure ordinal** of f.

But, computing closure ordinals in phase spaces is hard!

What if we say we approximate lfp and gfp by their ω-th approximation?

$$[\![\mu X.\varphi]\!] = \left(\bigcup_{n \geq 0} [\![\varphi^n(0)]\!]\right)^{\perp\perp} \qquad [\![\nu X.\varphi]\!] = \bigcap_{n \geq 0} [\![\varphi^n(\top)]\!]$$

This gives us an idea for new inference rules for fixpoints:

$$\frac{\vdash \Gamma, \overbrace{\varphi(\varphi(\cdots(\varphi}^{n}(0))\cdots)}{\vdash \Gamma, \mu X.\varphi}\,(\mu_\omega) \qquad \frac{\vdash \Gamma, \top \quad \vdash \Gamma, \varphi(\top) \quad \vdash \Gamma, \varphi(\varphi(\top)) \quad \ldots}{\vdash \Gamma, \nu X.\varphi}\,(\nu_\omega)$$

We call this system $\mu\mathsf{MALL}_\omega$.

Theorem

The new intepretation is sound and complete wrt μMALL$_\omega$.

Theorem

μMALL$_\omega$ *admits cuts.*

Advantage Completeness is easy since there is a (sort-of) subformula property. Cut admissibility can be proved using standard techniques from arithmetic.

Disadvantage Does not prove the same theorems as μMALL$^{\text{ind}}$.

- Is μMALL$^\infty$ Π^0_1-complete?

  Either give a Π^0_1 algorithm or improve the lower bound.
- How does one extend the truth semantics to μMALL$^\infty$?

  Soundness proofs for non-wellfounded calculi usually goes through finding a chain of countermodels that imply non-progression of an infinite branch. We do find the chain, but the non-progression is not clear.
- Compute closure ordinals of (classes of) μMALL formulas

  Computing closure ordinal of formulas as simple $a \parr x$ is difficult.
- Is $\mu\mathsf{MALL}_\omega \subsetneq \mu\mathsf{MALL}^{\mathsf{ind}}$?

Thank you!