Example

$$\text{restart } h\ n := \text{if } n > \underline{0} \text{ then } (\text{output } n; \text{restart } h\ (n - \underline{1})) \text{ else } h\ \underline{0}$$

$$\begin{aligned}\text{rsp}_1\ n := {} & \text{let } v = (\text{output } n; \text{input}()) * \underline{2} \text{ in} \\ & (\text{if } v \neq \underline{0} \text{ then rsp}_1 \text{ else restart rsp}_1)\ (v + n)\end{aligned}$$

$$\begin{aligned}\text{rsp}_2\ n := {} & \text{output } (\underline{2} * n); \\ & \text{let } v = \text{input}() \text{ in} \\ & \text{if } v = \underline{0} \text{ then restart rsp}_2\ (\underline{2} * n) \text{ else rsp}_2\ (v + n)\end{aligned}$$

Weak Simulation

$$
\begin{array}{lll}
e_1 \precsim e_2 & \text{iff} & (\forall q, e_1'.\ e_1 \xrightarrow{q} e_1' \implies \exists e_2'.\ e_2 \overset{q}{\rightsquigarrow} e_2' \wedge e_1' \precsim e_2') \\
& & \vee \\
& & (\forall v.\ e_1 = v \implies e_2 \overset{\tau}{\rightsquigarrow} v)
\end{array}
$$

$$
q \quad := \quad \tau \mid \text{in } n \mid \text{out } n
$$

$$
e \overset{q}{\rightsquigarrow} e' \quad \stackrel{\text{def}}{=} \quad \begin{cases} e \xrightarrow{\tau^*} e' & \text{if } q = \tau \\ e \xrightarrow{\tau^*}\xrightarrow{q}\xrightarrow{\tau^*} e' & \text{otherwise} \end{cases}
$$

Weak Simulation

$$\precsim \;=\; f_{\mathsf{sim}}(\precsim)$$

$$\begin{aligned} f_{\mathsf{sim}}(\precsim) \;=\; \{\, (e_1, e_2) \mid\; & (\forall q, e_1'.\ e_1 \xrightarrow{q} e_1' \implies \exists e_2'.\ e_2 \overset{q}{\rightsquigarrow} e_2' \wedge e_1' \precsim e_2') \\ & \vee \\ & (\forall v.\ e_1 = v \implies e_2 \overset{\tau}{\rightsquigarrow} v) \,\} \end{aligned}$$

$$q \;:=\; \tau \mid \textsf{in}\ n \mid \textsf{out}\ n$$

$$e \overset{q}{\rightsquigarrow} e' \;\overset{\text{def}}{=}\; \begin{cases} e \xrightarrow{\tau^*} e' & \text{if } q = \tau \\ e \xrightarrow{\tau^*}\xrightarrow{q}\xrightarrow{\tau^*} e' & \text{otherwise} \end{cases}$$

Weak Simulation

$$\precsim \;\stackrel{\text{def}}{=}\; \nu f_{\text{sim}}$$

$$\begin{aligned} f_{\text{sim}}(\precsim) = \{\, (e_1, e_2) \mid \\ & (\forall q, e_1'.\ e_1 \xrightarrow{q} e_1' \implies \exists e_2'.\ e_2 \overset{q}{\rightsquigarrow} e_2' \land e_1' \precsim e_2') \\ & \lor \\ & (\forall v.\ e_1 = v \implies e_2 \overset{\tau}{\rightsquigarrow} v) \,\} \end{aligned}$$

$$q \;:=\; \tau \mid \text{in } n \mid \text{out } n$$

$$e \overset{q}{\rightsquigarrow} e' \;\stackrel{\text{def}}{=}\; \begin{cases} e \xrightarrow{\tau^*} e' & \text{if } q = \tau \\ e \xrightarrow{\tau^*}\xrightarrow{q}\xrightarrow{\tau^*} e' & \text{otherwise} \end{cases}$$

Proof using Parameterized Coinduction

$$\text{restart } h\ n := \text{if } n > \underline{0} \text{ then } (\text{output } n; \text{restart } h\ (n - \underline{1})) \text{ else } h\ \underline{0}$$

$$\begin{aligned}\text{rsp}_1\ n := {} & \text{let } v = (\text{output } n; \text{input}()) * \underline{2} \text{ in} \\ & (\text{if } v \neq \underline{0} \text{ then rsp}_1 \text{ else restart rsp}_1)\ (v + n)\end{aligned}$$

$$\begin{aligned}\text{rsp}_2\ n := {} & \text{output } (\underline{2} * n); \\ & \text{let } v = \text{input}() \text{ in} \\ & \text{if } v = \underline{0} \text{ then restart rsp}_2\ (\underline{2} * n) \text{ else rsp}_2\ (v + n)\end{aligned}$$

$$\{\, (\text{rsp}_1\ \underline{2n}, \text{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\} \quad \subseteq \quad \nu f_{\text{sim}}$$

Proof using Parameterized Coinduction

$$\text{restart } h\ n := \text{if } n > \underline{0} \text{ then } (\text{output } n; \text{restart } h\ (n - \underline{1})) \text{ else } h\ \underline{0}$$

$$\begin{aligned}\text{rsp}_1\ n := {} & \text{let } v = (\text{output } n; \text{input}()) * \underline{2} \text{ in} \\ & (\text{if } v \neq \underline{0} \text{ then rsp}_1 \text{ else restart rsp}_1)\ (v + n)\end{aligned}$$

$$\begin{aligned}\text{rsp}_2\ n := {} & \text{output } (\underline{2} * n); \\ & \text{let } v = \text{input}() \text{ in} \\ & \text{if } v = \underline{0} \text{ then restart rsp}_2\ (\underline{2} * n) \text{ else rsp}_2\ (v + n)\end{aligned}$$

$$G_{\text{sim}}(A) = \nu(\lambda X.\, f_{\text{sim}}(A \cup X))$$

$$\{\, (\text{rsp}_1\ \underline{2n}, \text{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\} \quad \subseteq \quad G_{\text{sim}}(\emptyset)$$

Proof using Parameterized Coinduction

$$\text{restart } h\ n := \text{if } n > \underline{0} \text{ then } (\text{output } n; \text{restart } h\ (n - \underline{1})) \text{ else } h\ \underline{0}$$

$$\begin{aligned}\text{rsp}_1\ n := {} & \text{let } v = (\text{output } n; \text{input}()) * \underline{2} \text{ in} \\ & (\text{if } v \neq \underline{0} \text{ then rsp}_1 \text{ else restart rsp}_1)\ (v + n)\end{aligned}$$

$$\begin{aligned}\text{rsp}_2\ n := {} & \text{output } (\underline{2} * n); \\ & \text{let } v = \text{input}() \text{ in} \\ & \text{if } v = \underline{0} \text{ then restart rsp}_2\ (\underline{2} * n) \text{ else rsp}_2\ (v + n)\end{aligned}$$

$$G_{\text{sim}}(A) = \nu(\lambda X.\, f_{\text{sim}}(A \cup X))$$
$$R_0 = \{\, (\text{rsp}_1\ \underline{2n}, \text{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

$$R_0 \quad \subseteq \quad G_{\text{sim}}(\emptyset)$$

Proof using Parameterized Coinduction

restart $h\ n :=$ if $n > \underline{0}$ then (output n; restart $h\ (n - \underline{1})$) else $h\ \underline{0}$

$\mathsf{rsp}_1\ n :=$ let $v =$ (output n; input()) $* \underline{2}$ in
(if $v \neq \underline{0}$ then rsp_1 else restart rsp_1) $(v + n)$

$\mathsf{rsp}_2\ n :=$ output $(\underline{2} * n)$;
let $v =$ input() in
if $v = \underline{0}$ then restart $\mathsf{rsp}_2\ (\underline{2} * n)$ else $\mathsf{rsp}_2\ (v + n)$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

$$R_0 \quad \subseteq \quad G_{\mathsf{sim}}(R_0)$$

Proof using Parameterized Coinduction

$\text{restart } h\ n := \text{if } n > \underline{0} \text{ then } (\text{output } n; \text{restart } h\ (n - \underline{1})) \text{ else } h\ \underline{0}$

$$\begin{aligned}\text{rsp}_1\ n := {} & \text{let } v = (\text{output } n; \text{input}()) * \underline{2} \text{ in} \\ & (\text{if } v \neq \underline{0} \text{ then rsp}_1 \text{ else restart rsp}_1)\ (v + n)\end{aligned}$$

$$\begin{aligned}\text{rsp}_2\ n := {} & \text{output } (\underline{2} * n); \\ & \text{let } v = \text{input}() \text{ in} \\ & \text{if } v = \underline{0} \text{ then restart rsp}_2\ (\underline{2} * n) \text{ else rsp}_2\ (v + n)\end{aligned}$$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\text{rsp}_1\ \underline{2n}, \text{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

$$\{\, (\text{rsp}_1\ \underline{2n}, \text{rsp}_2\ \underline{n}) \,\} \quad \subseteq \quad G_{\mathsf{sim}}(R_0)$$

Proof using Parameterized Coinduction

restart $h\ n$:= if $n > \underline{0}$ then (output n; restart $h\ (n - \underline{1})$) else $h\ \underline{0}$

rsp$_1$ n := let v = (output n; input()) $* \underline{2}$ in
(if $v \neq \underline{0}$ then rsp$_1$ else restart rsp$_1$) $(v + n)$

rsp$_2$ n := output $(\underline{2} * n)$;
let v = input() in
if $v = \underline{0}$ then restart rsp$_2$ $(\underline{2} * n)$ else rsp$_2$ $(v + n)$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

$$\{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \,\} \quad \subseteq \quad f_{\mathsf{sim}}(R_0 \cup G_{\mathsf{sim}}(R_0))$$

Proof using Parameterized Coinduction

$\text{restart } h\ n := \text{if } n > \underline{0} \text{ then } (\text{output } n; \text{restart } h\ (n - \underline{1})) \text{ else } h\ \underline{0}$

$$\begin{aligned} e_1^1 := &\ \text{let } v = (\text{output } \underline{2n}; \text{input}()) * \underline{2} \text{ in} \\ &\ (\text{if } v \neq \underline{0} \text{ then rsp}_1 \text{ else restart rsp}_1)\ (v + \underline{2n}) \end{aligned}$$

$$\begin{aligned} e_2^1 := &\ \text{output } (\underline{2} * \underline{n}); \\ &\ \text{let } v = \text{input}() \text{ in} \\ &\ \text{if } v = \underline{0} \text{ then restart rsp}_2\ (\underline{2} * \underline{n}) \text{ else rsp}_2\ (v + \underline{n}) \end{aligned}$$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{ (\text{rsp}_1\ \underline{2n}, \text{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \}$$

$$\{ (\text{rsp}_1\ \underline{2n}, \text{rsp}_2\ \underline{n}) \} \quad \subseteq \quad f_{\mathsf{sim}}(R_0 \cup G_{\mathsf{sim}}(R_0))$$

Proof using Parameterized Coinduction

restart h n := if $n > \underline{0}$ then (output n; restart h $(n - \underline{1})$) else h $\underline{0}$

e_1^1 := let v = (output $\underline{2n}$; input()) $* \underline{2}$ in
(if $v \neq \underline{0}$ then rsp_1 else restart rsp_1) $(v + \underline{2n})$

e_2^1 := output $(\underline{2} * \underline{n})$;
let v = input() in
if $v = \underline{0}$ then restart rsp_2 $(\underline{2} * \underline{n})$ else rsp_2 $(v + \underline{n})$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

$$\{\, (e_1^1, e_2^1) \,\} \quad \subseteq \quad R_0 \cup G_{\mathsf{sim}}(R_0)$$

Proof using Parameterized Coinduction

$$\text{restart } h\ n := \text{if } n > \underline{0} \text{ then } (\text{output } n; \text{restart } h\ (n - \underline{1})) \text{ else } h\ \underline{0}$$

$$\begin{aligned} e_1^1 &:= \text{let } v = (\text{output } \underline{2n}; \text{input}()) * \underline{2} \text{ in} \\ &\quad\ (\text{if } v \neq \underline{0} \text{ then rsp}_1 \text{ else restart rsp}_1)\ (v + \underline{2n}) \end{aligned}$$

$$\begin{aligned} e_2^1 &:= \text{output } (\underline{2} * \underline{n}); \\ &\quad\ \text{let } v = \text{input}() \text{ in} \\ &\quad\ \text{if } v = \underline{0} \text{ then restart rsp}_2\ (\underline{2} * \underline{n}) \text{ else rsp}_2\ (v + \underline{n}) \end{aligned}$$

$$G_{\text{sim}}(A) = \nu(\lambda X.\, f_{\text{sim}}(A \cup X))$$
$$R_0 = \{\, (\text{rsp}_1\ \underline{2n}, \text{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

$$\{\, (e_1^1, e_2^1) \,\} \quad \subseteq \quad G_{\text{sim}}(R_0)$$

Proof using Parameterized Coinduction

restart $h\ n :=$ if $n > \underline{0}$ then (output n; restart $h\ (n - \underline{1})$) else $h\ \underline{0}$

$e_1^2 :=$ let $v = (\langle\rangle; \text{input}()) * \underline{2}$ in (if $v \neq \underline{0}$ then rsp_1 else restart rsp_1) $(v + \underline{2n})$

out $2n$

$e_2^1 :=$ output $(\underline{2} * \underline{n})$;
let $v = \text{input}()$ in
if $v = \underline{0}$ then restart $\text{rsp}_2\ (\underline{2} * \underline{n})$ else $\text{rsp}_2\ (v + \underline{n})$

$$G_{\text{sim}}(A) = \nu(\lambda X.\, f_{\text{sim}}(A \cup X))$$
$$R_0 = \{\, (\text{rsp}_1\ \underline{2n}, \text{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

$$\{\, (e_1^1, e_2^1) \,\} \quad \subseteq \quad G_{\text{sim}}(R_0)$$

Proof using Parameterized Coinduction

restart $h\ n$:= if $n > \underline{0}$ then (output n; restart $h\ (n - \underline{1})$) else $h\ \underline{0}$

e_1^2 := let $v = (\langle\rangle; \text{input}()) * \underline{2}$ in (if $v \neq \underline{0}$ then rsp_1 else restart rsp_1) $(v + \underline{2n})$ — out $2n$

e_2^2 := $\langle\rangle$; let v = input() in if $v = \underline{0}$ then restart rsp_2 $(\underline{2} * \underline{n})$ else rsp_2 $(v + \underline{n})$ — out $2n$

$$G_{\text{sim}}(A) = \nu(\lambda X.\, f_{\text{sim}}(A \cup X))$$
$$R_0 = \{\, (\text{rsp}_1\ \underline{2n}, \text{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

$$\{\, (e_1^1, e_2^1) \,\} \quad \subseteq \quad G_{\text{sim}}(R_0)$$

Proof using Parameterized Coinduction

$$\text{restart } h\ n := \text{if } n > \underline{0} \text{ then } (\text{output } n; \text{restart } h\ (n - \underline{1})) \text{ else } h\ \underline{0}$$

$$\begin{aligned} e_1^2 := {} & \text{let } v = (\langle\rangle; \text{input}()) * \underline{2} \text{ in} \\ & (\text{if } v \neq \underline{0} \text{ then } \text{rsp}_1 \text{ else restart } \text{rsp}_1)\ (v + \underline{2n}) \end{aligned} \qquad \boxed{\text{out } 2n}$$

$$\begin{aligned} e_2^2 := {} & \langle\rangle; \\ & \text{let } v = \text{input}() \text{ in} \\ & \text{if } v = \underline{0} \text{ then restart } \text{rsp}_2\ (\underline{2} * \underline{n}) \text{ else } \text{rsp}_2\ (v + \underline{n}) \end{aligned} \qquad \boxed{\text{out } 2n}$$

$$G_{\text{sim}}(A) = \nu(\lambda X.\, f_{\text{sim}}(A \cup X))$$
$$R_0 = \{ (\text{rsp}_1\ \underline{2n}, \text{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \}$$

$$\{ (e_1^2, e_2^2) \} \quad \subseteq \quad G_{\text{sim}}(R_0)$$

Proof using Parameterized Coinduction

restart $h\ n$:= if $n > \underline{0}$ then (output n; restart $h\ (n - \underline{1})$) else $h\ \underline{0}$

e_1^3 := let $v = \underline{m} * \underline{2}$ in (if $v \neq \underline{0}$ then rsp_1 else restart rsp_1) $(v + \underline{2n})$ — [in m]

e_2^2 := $\langle\rangle$; let v = input() in if $v = \underline{0}$ then restart rsp_2 $(\underline{2} * \underline{n})$ else rsp_2 $(v + \underline{n})$ — [out $2n$]

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

$$\{\, (e_1^2, e_2^2) \,\} \quad \subseteq \quad G_{\mathsf{sim}}(R_0)$$

Proof using Parameterized Coinduction

restart h n := if $n > \underline{0}$ then (output n; restart h $(n - \underline{1})$) else h $\underline{0}$

e_1^3 := let $v = \underline{m} * \underline{2}$ in
(if $v \neq \underline{0}$ then rsp_1 else restart rsp_1) $(v + \underline{2n})$ — in m

e_2^3 := — in m
let $v = \underline{m}$ in
if $v = \underline{0}$ then restart rsp_2 $(\underline{2} * \underline{n})$ else rsp_2 $(v + \underline{n})$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

$$\{\, (e_1^2, e_2^2) \,\} \quad \subseteq \quad G_{\mathsf{sim}}(R_0)$$

Proof using Parameterized Coinduction

$$\text{restart } h\ n := \text{if } n > \underline{0} \text{ then } (\text{output } n; \text{restart } h\ (n - \underline{1})) \text{ else } h\ \underline{0}$$

$$e_1^3 := \text{let } v = \underline{m} * \underline{2} \text{ in } (\text{if } v \neq \underline{0} \text{ then } \mathsf{rsp}_1 \text{ else restart } \mathsf{rsp}_1)\ (v + \underline{2n}) \qquad \boxed{\text{in } m}$$

$$e_2^3 := \text{let } v = \underline{m} \text{ in if } v = \underline{0} \text{ then restart } \mathsf{rsp}_2\ (\underline{2} * \underline{n}) \text{ else } \mathsf{rsp}_2\ (v + \underline{n}) \qquad \boxed{\text{in } m}$$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$

$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

$$\{\, (e_1^3, e_2^3) \,\} \quad \subseteq \quad G_{\mathsf{sim}}(R_0)$$

Proof using Parameterized Coinduction

$$\text{restart } h\ n := \text{if } n > \underline{0} \text{ then } (\text{output } n; \text{restart } h\ (n - \underline{1})) \text{ else } h\ \underline{0}$$

$$e_1^4 :=$$
$$(\text{if } \underline{2m} \neq \underline{0} \text{ then rsp}_1 \text{ else restart rsp}_1)\ (\underline{2m} + \underline{2n})$$

$$e_2^3 :=$$
$$\text{let } v = \underline{m} \text{ in}$$
$$\text{if } v = \underline{0} \text{ then restart rsp}_2\ (\underline{2} * \underline{n}) \text{ else rsp}_2\ (v + \underline{n})$$

$\boxed{\text{in } m}$

$$G_{\text{sim}}(A) = \nu(\lambda X.\, f_{\text{sim}}(A \cup X))$$
$$R_0 = \{\, (\text{rsp}_1\ \underline{2n}, \text{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

$$\{\, (e_1^3, e_2^3) \,\} \quad \subseteq \quad G_{\text{sim}}(R_0)$$

Proof using Parameterized Coinduction

restart $h\ n :=$ if $n > \underline{0}$ then (output n; restart $h\ (n - \underline{1})$) else $h\ \underline{0}$

$e_1^4 :=$

(if $\underline{2m} \neq \underline{0}$ then rsp_1 else restart rsp_1) $(\underline{2m} + \underline{2n})$

$e_2^4 :=$

if $\underline{m} = \underline{0}$ then restart $\mathsf{rsp}_2\ (\underline{2} * \underline{n})$ else $\mathsf{rsp}_2\ (\underline{m} + \underline{n})$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

$$\{\, (e_1^3, e_2^3) \,\} \quad \subseteq \quad G_{\mathsf{sim}}(R_0)$$

Proof using Parameterized Coinduction

restart $h\ n :=$ if $n > \underline{0}$ then (output n; restart $h\ (n - \underline{1})$) else $h\ \underline{0}$

$e_1^4 :=$

(if $\underline{2m} \neq \underline{0}$ then rsp_1 else restart rsp_1) $(\underline{2m} + \underline{2n})$

$e_2^4 :=$

if $\underline{m} = \underline{0}$ then restart $\mathsf{rsp}_2\ (\underline{2} * \underline{n})$ else $\mathsf{rsp}_2\ (\underline{m} + \underline{n})$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

$$\{\, (e_1^4, e_2^4) \,\} \quad \subseteq \quad G_{\mathsf{sim}}(R_0)$$

Proof using Parameterized Coinduction

$$\text{restart } h\ n := \text{if } n > \underline{0} \text{ then } (\text{output } n; \text{restart } h\ (n - \underline{1})) \text{ else } h\ \underline{0}$$

$$e_1^4 :=$$
$$(\text{if } \underline{2m} \neq \underline{0} \text{ then } \mathsf{rsp}_1 \text{ else restart } \mathsf{rsp}_1)\ (\underline{2m} + \underline{2n})$$

$$e_2^4 :=$$
$$\text{if } \underline{m} = \underline{0} \text{ then restart } \mathsf{rsp}_2\ (\underline{2} * \underline{n}) \text{ else } \mathsf{rsp}_2\ (\underline{m} + \underline{n})$$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{ (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \}$$

When $m \neq 0$:

$$\{ (e_1^4, e_2^4) \} \quad \subseteq \quad G_{\mathsf{sim}}(R_0)$$

Proof using Parameterized Coinduction

$\text{restart}\ h\ n := \text{if}\ n > \underline{0}\ \text{then}\ (\text{output}\ n; \text{restart}\ h\ (n - \underline{1}))\ \text{else}\ h\ \underline{0}$

$e_1^5 :=$

$\text{rsp}_1\ (\underline{2m + 2n})$

$e_2^4 :=$

$\text{if}\ \underline{m} = \underline{0}\ \text{then restart}\ \text{rsp}_2\ (\underline{2} * \underline{n})\ \text{else}\ \text{rsp}_2\ (\underline{m} + \underline{n})$

$$G_{\text{sim}}(A) = \nu(\lambda X.\, f_{\text{sim}}(A \cup X))$$
$$R_0 = \{\, (\text{rsp}_1\ \underline{2n}, \text{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

When $m \neq 0$:

$$\{\, (e_1^4, e_2^4) \,\} \quad \subseteq \quad G_{\text{sim}}(R_0)$$

Proof using Parameterized Coinduction

restart $h\ n :=$ if $n > \underline{0}$ then (output n; restart $h\ (n - \underline{1})$) else $h\ \underline{0}$

$e_1^5 :=$

$\quad \mathsf{rsp}_1\ (\underline{2m + 2n})$

$e_2^5 :=$

$\quad \mathsf{rsp}_2\ (\underline{m + n})$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

When $m \neq 0$:

$$\{\, (e_1^4, e_2^4) \,\} \quad \subseteq \quad G_{\mathsf{sim}}(R_0)$$

Proof using Parameterized Coinduction

$\text{restart } h\ n := \text{if } n > \underline{0} \text{ then } (\text{output } n; \text{restart } h\ (n - \underline{1})) \text{ else } h\ \underline{0}$

$e_1^5 :=$

$\quad \mathsf{rsp}_1\ (\underline{2m + 2n})$

$e_2^5 :=$

$\quad \mathsf{rsp}_2\ (\underline{m + n})$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

When $m \neq 0$:

$$\{\, (e_1^4, e_2^4) \,\} \quad \subseteq \quad f_{\mathsf{sim}}(R_0 \cup G_{\mathsf{sim}}(R_0))$$

Proof using Parameterized Coinduction

restart $h\ n :=$ if $n > \underline{0}$ then (output n; restart $h\ (n - \underline{1})$) else $h\ \underline{0}$

$e_1^5 :=$

$\mathsf{rsp}_1\ (\underline{2m + 2n})$

$e_2^5 :=$

$\mathsf{rsp}_2\ (\underline{m + n})$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

When $m \neq 0$:

$$\{\, (e_1^5, e_2^5) \,\} \quad \subseteq \quad R_0 \cup G_{\mathsf{sim}}(R_0)$$

Proof using Parameterized Coinduction

restart $h\ n :=$ if $n > \underline{0}$ then (output n; restart $h\ (n - \underline{1})$) else $h\ \underline{0}$

$e_1^5 :=$

$\mathsf{rsp}_1\ (\underline{2m + 2n})$

$e_2^5 :=$

$\mathsf{rsp}_2\ (\underline{m + n})$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

When $m \neq 0$:

$$\{\, (e_1^5, e_2^5) \,\} \quad \subseteq \quad R_0 \cup G_{\mathsf{sim}}(R_0) \quad \checkmark$$

Proof using Parameterized Coinduction

restart $h\ n :=$ if $n > \underline{0}$ then (output n; restart $h\ (n - \underline{1})$) else $h\ \underline{0}$

$e_1^4 :=$

(if $\underline{2m} \neq \underline{0}$ then rsp_1 else restart rsp_1) $(\underline{2m} + \underline{2n})$

$e_2^4 :=$

if $\underline{m} = \underline{0}$ then restart rsp_2 $(\underline{2} * \underline{n})$ else rsp_2 $(\underline{m} + \underline{n})$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\text{rsp}_1\ \underline{2n}, \text{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

When $m = 0$:

$$\{\, (e_1^4, e_2^4) \,\} \quad \subseteq \quad G_{\mathsf{sim}}(R_0)$$

Proof using Parameterized Coinduction

$\text{restart } h\ n := \text{if } n > \underline{0} \text{ then } (\text{output } n; \text{restart } h\ (n - \underline{1})) \text{ else } h\ \underline{0}$

$e_1^5 :=$

$\text{restart rsp}_1\ \underline{2n}$

$e_2^4 :=$

$\text{if } \underline{m} = \underline{0} \text{ then restart rsp}_2\ (\underline{2} * \underline{n}) \text{ else rsp}_2\ (\underline{m} + \underline{n})$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\text{rsp}_1\ \underline{2n}, \text{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

When $m = 0$:

$$\{\, (e_1^4, e_2^4) \,\} \quad \subseteq \quad G_{\mathsf{sim}}(R_0)$$

Proof using Parameterized Coinduction

restart $h\ n :=$ if $n > \underline{0}$ then (output n; restart $h\ (n - \underline{1})$) else $h\ \underline{0}$

$e_1^5 :=$

restart $\mathsf{rsp}_1\ \underline{2n}$

$e_2^5 :=$

restart $\mathsf{rsp}_2\ \underline{2n}$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

When $m = 0$:

$$\{\, (e_1^4, e_2^4) \,\} \quad \subseteq \quad G_{\mathsf{sim}}(R_0)$$

Proof using Parameterized Coinduction

restart h n := if $n > \underline{0}$ then (output n; restart h $(n - \underline{1})$) else h $\underline{0}$

$e_1^5 :=$

restart rsp_1 $\underline{2n}$

$e_2^5 :=$

restart rsp_2 $\underline{2n}$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

$$\{\, (e_1^5, e_2^5) \,\} \quad \subseteq \quad G_{\mathsf{sim}}(R_0)$$

Proof using Parameterized Coinduction

restart $h\ n :=$ if $n > \underline{0}$ then (output n; restart $h\ (n - \underline{1})$) else $h\ \underline{0}$

$e_1^5 :=$

restart $\mathsf{rsp}_1\ \underline{2n}$

$e_2^5 :=$

restart $\mathsf{rsp}_2\ \underline{2n}$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$
$$R_1 = \{\, (\text{restart } \mathsf{rsp}_1\ \underline{n}, \text{restart } \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

$$\{\, (e_1^5, e_2^5) \,\} \quad \subseteq \quad G_{\mathsf{sim}}(R_0)$$

Proof using Parameterized Coinduction

$\text{restart}\ h\ n := \text{if}\ n > \underline{0}\ \text{then}\ (\text{output}\ n; \text{restart}\ h\ (n - \underline{1}))\ \text{else}\ h\ \underline{0}$

$e_1^5 :=$

$\text{restart rsp}_1\ \underline{2n}$

$e_2^5 :=$

$\text{restart rsp}_2\ \underline{2n}$

$$G_{\text{sim}}(A) = \nu(\lambda X.\, f_{\text{sim}}(A \cup X))$$
$$R_0 = \{\, (\text{rsp}_1\ \underline{2n}, \text{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$
$$R_1 = \{\, (\text{restart rsp}_1\ \underline{n}, \text{restart rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

$$R_1 \quad \subseteq \quad G_{\text{sim}}(R_0)$$

Proof using Parameterized Coinduction

restart h n := if $n > \underline{0}$ then (output n; restart h $(n - \underline{1})$) else h $\underline{0}$

$$
\begin{aligned}
G_{\mathsf{sim}}(A) &= \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X)) \\
R_0 &= \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\} \\
R_1 &= \{\, (\mathsf{restart}\ \mathsf{rsp}_1\ \underline{n}, \mathsf{restart}\ \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}
\end{aligned}
$$

$$
R_1 \quad \subseteq \quad G_{\mathsf{sim}}(R_0 \cup R_1)
$$

Proof using Parameterized Coinduction

restart h n := if $n > \underline{0}$ then (output n; restart h $(n - \underline{1})$) else h $\underline{0}$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$
$$R_1 = \{\, (\mathsf{restart}\ \mathsf{rsp}_1\ \underline{n}, \mathsf{restart}\ \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

$$\{\, (\mathsf{restart}\ \mathsf{rsp}_1\ \underline{n}, \mathsf{restart}\ \mathsf{rsp}_2\ \underline{n}) \,\} \quad \subseteq \quad G_{\mathsf{sim}}(R_0 \cup R_1)$$

Proof using Parameterized Coinduction

$$\text{restart } h\ n := \text{if } n > \underline{0} \text{ then } (\text{output } n; \text{restart } h\ (n - \underline{1})) \text{ else } h\ \underline{0}$$

$$e_1^1 := \text{if } \underline{n} > \underline{0} \text{ then } (\text{output } \underline{n}; \text{restart rsp}_1\ (\underline{n} - \underline{1})) \text{ else rsp}_1\ \underline{0}$$

$$e_2^1 := \text{if } \underline{n} > \underline{0} \text{ then } (\text{output } \underline{n}; \text{restart rsp}_2\ (\underline{n} - \underline{1})) \text{ else rsp}_2\ \underline{0}$$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{ (\text{rsp}_1\ \underline{2n}, \text{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \}$$
$$R_1 = \{ (\text{restart rsp}_1\ \underline{n}, \text{restart rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \}$$

$$\{ (\text{restart rsp}_1\ \underline{n}, \text{restart rsp}_2\ \underline{n}) \} \quad \subseteq \quad G_{\mathsf{sim}}(R_0 \cup R_1)$$

Proof using Parameterized Coinduction

$$\text{restart } h\ n := \text{if } n > \underline{0} \text{ then } (\text{output } n; \text{restart } h\ (n - \underline{1})) \text{ else } h\ \underline{0}$$

$$e_1^1 := \text{if } \underline{n} > \underline{0} \text{ then } (\text{output } \underline{n}; \text{restart } \text{rsp}_1\ (\underline{n} - \underline{1})) \text{ else } \text{rsp}_1\ \underline{0}$$

$$e_2^1 := \text{if } \underline{n} > \underline{0} \text{ then } (\text{output } \underline{n}; \text{restart } \text{rsp}_2\ (\underline{n} - \underline{1})) \text{ else } \text{rsp}_2\ \underline{0}$$

$$G_{\text{sim}}(A) = \nu(\lambda X.\, f_{\text{sim}}(A \cup X))$$
$$R_0 = \{\, (\text{rsp}_1\ \underline{2n}, \text{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$
$$R_1 = \{\, (\text{restart } \text{rsp}_1\ \underline{n}, \text{restart } \text{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

$$\{\, (e_1^1, e_2^1) \,\} \quad \subseteq \quad G_{\text{sim}}(R_0 \cup R_1)$$

Proof using Parameterized Coinduction

restart h n := if $n > \underline{0}$ then (output n; restart h $(n - \underline{1})$) else h $\underline{0}$

e_1^1 := if $\underline{n} > \underline{0}$ then (output $\underline{n}$; restart rsp_1 $(\underline{n} - \underline{1})$) else rsp_1 $\underline{0}$

e_2^1 := if $\underline{n} > \underline{0}$ then (output $\underline{n}$; restart rsp_2 $(\underline{n} - \underline{1})$) else rsp_2 $\underline{0}$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$
$$R_1 = \{\, (\mathsf{restart}\ \mathsf{rsp}_1\ \underline{n}, \mathsf{restart}\ \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

When $n \le 0$:

$$\{\, (e_1^1, e_2^1) \,\} \quad \subseteq \quad G_{\mathsf{sim}}(R_0 \cup R_1)$$

Proof using Parameterized Coinduction

restart $h\ n :=$ if $n > \underline{0}$ then (output n; restart $h\ (n - \underline{1})$) else $h\ \underline{0}$

$e_1^2 :=$ rsp$_1$ $\underline{0}$

$e_2^2 :=$ rsp$_2$ $\underline{0}$

$$G_{\text{sim}}(A) = \nu(\lambda X.\, f_{\text{sim}}(A \cup X))$$
$$R_0 = \{\, (\text{rsp}_1\ \underline{2n}, \text{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$
$$R_1 = \{\, (\text{restart rsp}_1\ \underline{n}, \text{restart rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

When $n \leq 0$:

$$\{\, (e_1^1, e_2^1) \,\} \quad \subseteq \quad G_{\text{sim}}(R_0 \cup R_1)$$

Proof using Parameterized Coinduction

restart $h\ n :=$ if $n > \underline{0}$ then (output n; restart $h\ (n - \underline{1})$) else $h\ \underline{0}$

$e_1^2 :=$ rsp$_1\ \underline{0}$

$e_2^2 :=$ rsp$_2\ \underline{0}$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$
$$R_1 = \{\, (\mathsf{restart\ rsp}_1\ \underline{n}, \mathsf{restart\ rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

When $n \leq 0$:

$$\{\, (e_1^1, e_2^1) \,\} \quad \subseteq \quad f_{\mathsf{sim}}(R_0 \cup R_1 \cup G_{\mathsf{sim}}(R_0 \cup R_1))$$

Proof using Parameterized Coinduction

restart $h\ n :=$ if $n > \underline{0}$ then (output n; restart $h\ (n - \underline{1})$) else $h\ \underline{0}$

$e_1^2 :=$ rsp$_1$ $\underline{0}$

$e_2^2 :=$ rsp$_2$ $\underline{0}$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$
$$R_1 = \{\, (\mathsf{restart}\ \mathsf{rsp}_1\ \underline{n}, \mathsf{restart}\ \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

When $n \le 0$:

$$\{\, (e_1^2, e_2^2) \,\} \quad \subseteq \quad R_0 \cup R_1 \cup G_{\mathsf{sim}}(R_0 \cup R_1)$$

Proof using Parameterized Coinduction

restart $h\ n :=$ if $n > \underline{0}$ then (output n; restart $h\ (n - \underline{1})$) else $h\ \underline{0}$

$e_1^2 :=$ rsp$_1\ \underline{0}$

$e_2^2 :=$ rsp$_2\ \underline{0}$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$
$$R_1 = \{\, (\mathsf{restart}\ \mathsf{rsp}_1\ \underline{n}, \mathsf{restart}\ \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

When $n \leq 0$:

$$\{\, (e_1^2, e_2^2) \,\} \quad \subseteq \quad R_0 \cup R_1 \cup G_{\mathsf{sim}}(R_0 \cup R_1) \quad \checkmark$$

Proof using Parameterized Coinduction

$$\text{restart } h\ n := \text{if } n > \underline{0} \text{ then } (\text{output } n; \text{restart } h\ (n - \underline{1})) \text{ else } h\ \underline{0}$$

$$e_1^1 := \text{if } \underline{n} > \underline{0} \text{ then } (\text{output } \underline{n}; \text{restart rsp}_1\ (\underline{n} - \underline{1})) \text{ else rsp}_1\ \underline{0}$$

$$e_2^1 := \text{if } \underline{n} > \underline{0} \text{ then } (\text{output } \underline{n}; \text{restart rsp}_2\ (\underline{n} - \underline{1})) \text{ else rsp}_2\ \underline{0}$$

$$G_{\text{sim}}(A) = \nu(\lambda X.\, f_{\text{sim}}(A \cup X))$$
$$R_0 = \{\, (\text{rsp}_1\ \underline{2n}, \text{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$
$$R_1 = \{\, (\text{restart rsp}_1\ \underline{n}, \text{restart rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

When $n > 0$:

$$\{\, (e_1^1, e_2^1) \,\} \quad \subseteq \quad G_{\text{sim}}(R_0 \cup R_1)$$

Proof using Parameterized Coinduction

restart $h\ n$:= if $n > \underline{0}$ then (output n; restart $h\ (n - \underline{1})$) else $h\ \underline{0}$

e_1^2 := output $\underline{n}$; restart $\mathsf{rsp}_1\ (\underline{n} - \underline{1})$

e_2^2 := output $\underline{n}$; restart $\mathsf{rsp}_2\ (\underline{n} - \underline{1})$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$
$$R_1 = \{\, (\mathsf{restart}\ \mathsf{rsp}_1\ \underline{n}, \mathsf{restart}\ \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

When $n > 0$:

$$\{\, (e_1^1, e_2^1) \,\} \quad \subseteq \quad G_{\mathsf{sim}}(R_0 \cup R_1)$$

Proof using Parameterized Coinduction

restart $h\ n :=$ if $n > \underline{0}$ then (output n; restart $h\ (n - \underline{1})$) else $h\ \underline{0}$

$e_1^2 :=$ output $\underline{n}$; restart $\mathsf{rsp}_1\ (\underline{n} - \underline{1})$

$e_2^2 :=$ output $\underline{n}$; restart $\mathsf{rsp}_2\ (\underline{n} - \underline{1})$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$
$$R_1 = \{\, (\text{restart } \mathsf{rsp}_1\ \underline{n}, \text{restart } \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

When $n > 0$:

$$\{\, (e_1^2, e_2^2) \,\} \quad \subseteq \quad G_{\mathsf{sim}}(R_0 \cup R_1)$$

Proof using Parameterized Coinduction

restart h n := if $n > \underline{0}$ then (output n; restart h $(n - \underline{1})$) else h $\underline{0}$

e_1^3 := restart rsp_1 $\underline{n-1}$ | out n

e_2^3 := restart rsp_2 $\underline{n-1}$ | out n

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$
$$R_1 = \{\, (\text{restart } \mathsf{rsp}_1\ \underline{n}, \text{restart } \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

When $n > 0$:

$$\{\, (e_1^2, e_2^2) \,\} \quad \subseteq \quad G_{\mathsf{sim}}(R_0 \cup R_1)$$

Proof using Parameterized Coinduction

restart h n := if $n > \underline{0}$ then (output n; restart h $(n - \underline{1})$) else h $\underline{0}$

e_1^3 := restart rsp_1 $\underline{n-1}$ | out n

e_2^3 := restart rsp_2 $\underline{n-1}$ | out n

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$
$$R_1 = \{\, (\text{restart } \mathsf{rsp}_1\ \underline{n}, \text{restart } \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

When $n > 0$:

$$\{\, (e_1^2, e_2^2) \,\} \quad \subseteq \quad f_{\mathsf{sim}}(R_0 \cup R_1 \cup G_{\mathsf{sim}}(R_0 \cup R_1))$$

Proof using Parameterized Coinduction

restart h n := if $n > \underline{0}$ then (output n; restart h $(n - \underline{1})$) else h $\underline{0}$

e_1^3 := restart rsp_1 $\underline{n-1}$ | out n

e_2^3 := restart rsp_2 $\underline{n-1}$ | out n

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$
$$R_1 = \{\, (\text{restart } \mathsf{rsp}_1\ \underline{n}, \text{restart } \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

When $n > 0$:

$$\{\, (e_1^3, e_2^3) \,\} \quad \subseteq \quad R_0 \cup R_1 \cup G_{\mathsf{sim}}(R_0 \cup R_1)$$

Proof using Parameterized Coinduction

restart h n $:=$ if $n > \underline{0}$ then (output n; restart h $(n - \underline{1})$) else h $\underline{0}$

$e_1^3 :=$ restart rsp_1 $\underline{n-1}$ $\boxed{\text{out } n}$

$e_2^3 :=$ restart rsp_2 $\underline{n-1}$ $\boxed{\text{out } n}$

$$G_{\mathsf{sim}}(A) = \nu(\lambda X.\, f_{\mathsf{sim}}(A \cup X))$$
$$R_0 = \{\, (\mathsf{rsp}_1\ \underline{2n}, \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$
$$R_1 = \{\, (\text{restart } \mathsf{rsp}_1\ \underline{n}, \text{restart } \mathsf{rsp}_2\ \underline{n}) \mid n \in \mathbb{N} \,\}$$

When $n > 0$:

$$\{\, (e_1^3, e_2^3) \,\} \quad \subseteq \quad R_0 \cup R_1 \cup G_{\mathsf{sim}}(R_0 \cup R_1) \quad \checkmark$$