Coalgebra: applications in automata theory and programming language design

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Radboud Universiteit Nijmegen and CWI

Chocola meeting, 16 May 2013

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Specify and reason about systems.

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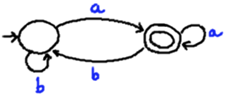
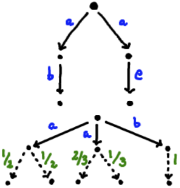
state-machines
e.g. DFA, LTS, PA





Coalgebra

Specify and **reason** about **systems**.

Syntax
RE, CCS, . . .

$b^*a(b^*a)^*$

$a.b.0 + a.c.0$

$a.(\frac{1}{2}\cdot 0 \oplus \frac{1}{2}\cdot 0) + \cdots$

state-machines
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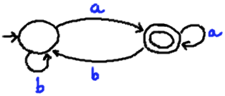

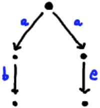

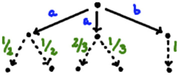

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Specify and **reason** about **systems**.

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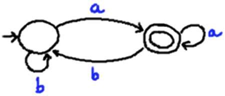
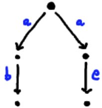
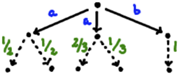
$b^*a(b^*a)^*$	$1 + a\,a^* = a^*$	
$a.b.0 + a.c.0$	$P + 0 = P$	
$a.(\frac{1}{2}\cdot 0 \oplus \frac{1}{2}\cdot 0) + \cdots$	$p.P \oplus p'.P = (p+p').P$	

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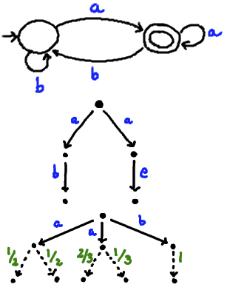
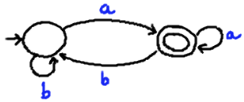
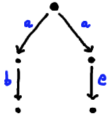


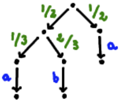
Can we do all of this uniformly in a single framework?

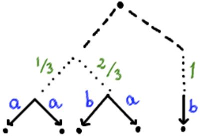
What do this things have in common?

$$(S, t : S \to 2 \times S^A)$$









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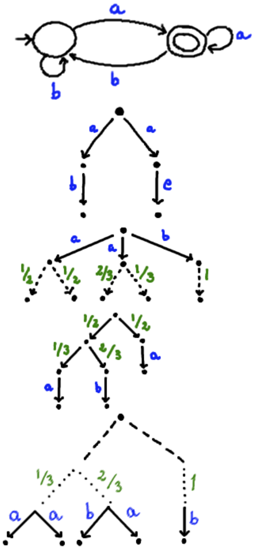

$(S, t : S \to 2 \times S^A)$

$(S, t : S \to \mathcal{P}S^A)$

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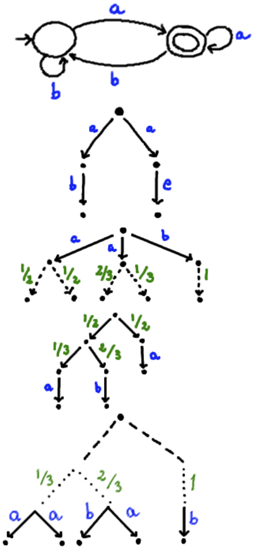

$$(S, t : S \to 2 \times S^A)$$

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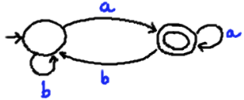

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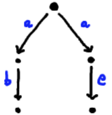

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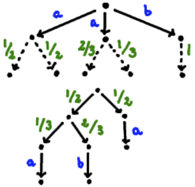

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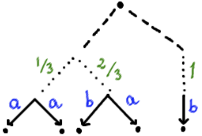

$$(S, t : S \to \mathcal{D}_\omega(S) + (A \times S) + 1)$$

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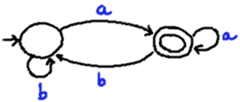

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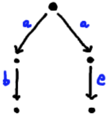

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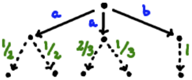

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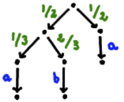

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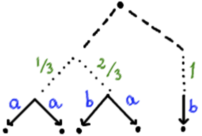

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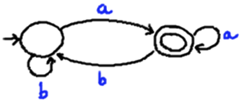

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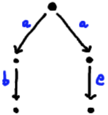

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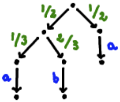

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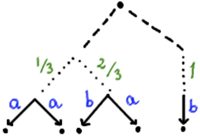

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$(S, t : S \to TS)$

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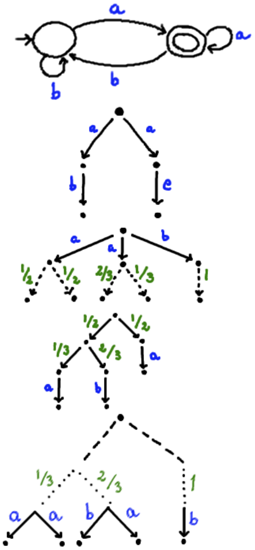

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$(S, t : S \to \mathcal{P}(\mathcal{D}_\omega(\mathcal{P}S)^A))$

$(S, t : S \to TS)$ T-coalgebras

The power of T

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   E.g. $T = 2 \times (-)^A$: languages over A – 2^{A^*}
3. set of expressions describing finite systems
4. axioms to prove bisimulation equivalence of expressions

1 + 2 are classic coalgebra; 3 + 4 are recent work.

How about algorithms?

- Coalgebra has found its place in the semantic side of the world: operational/denotational semantics, logics, . . .
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Brzozowski's algorithm (co)algebraically

Motivation

- duality between reachability and observability (Arbib and Manes 1975): beautiful, not very well-known.
- combined use of algebra and coalgebra.
- our understanding of automata is still very limited; cf. recent research: universal automata, àtomata, weighted automata (Sakarovitch, Brzozowski, . . .)

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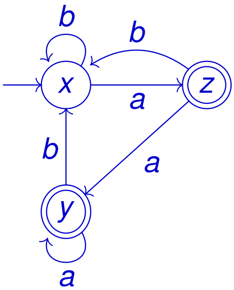
- joint work with Bonchi, Bonsangue, Rutten (Dexter's festschrift 2012)

Brzozowski algorithm (by example)



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- final states: y and z
- $L(x) \;=\; \{a, b\}^* \, a$

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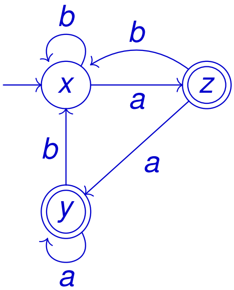
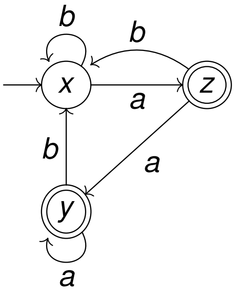


- initial state: x
- final states: y and z
- $L(x) \ = \ \{a, b\}^* \, a$
- X is reachable but not minimal: $L(y) \ = \ \varepsilon + \{a, b\}^* \, a \ = \ L(z)$

Reversing the automaton: $rev(X)$

$X =$



Reversing the automaton: *rev*(X)

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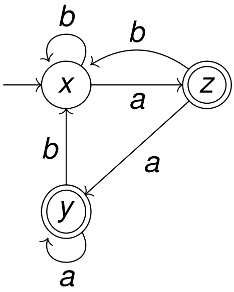

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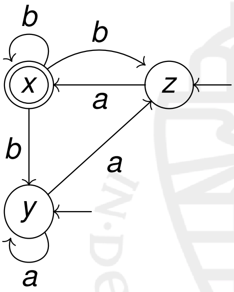

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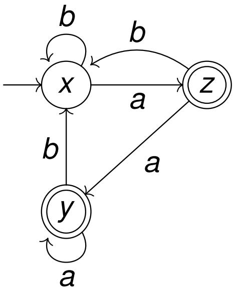

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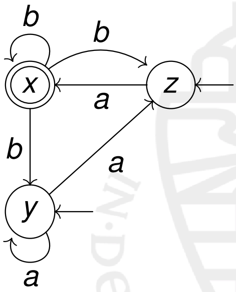

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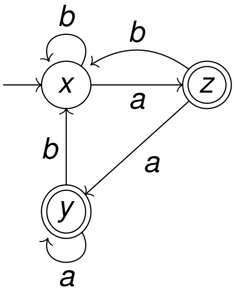

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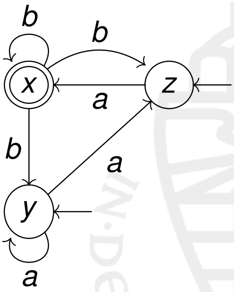
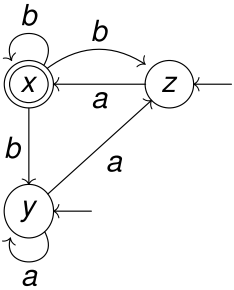


- transitions are reversed
- initial states $\Leftrightarrow$ final states
- $rev(X)$ is non-deterministic

Making it deterministic again: $det(rev(X))$



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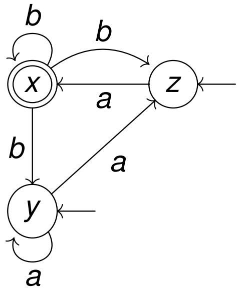

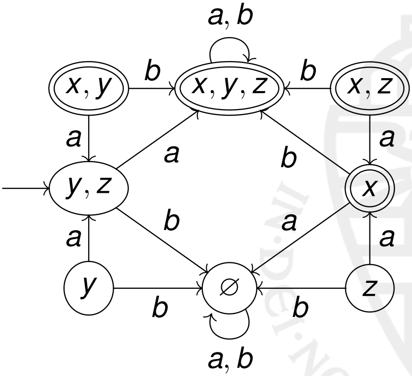

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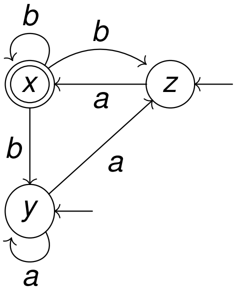
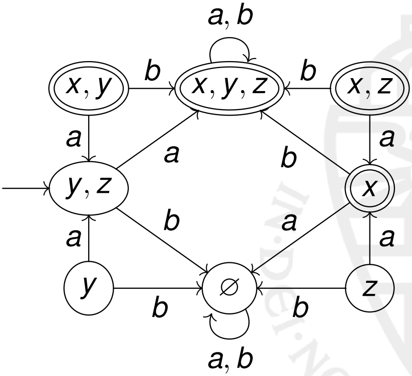
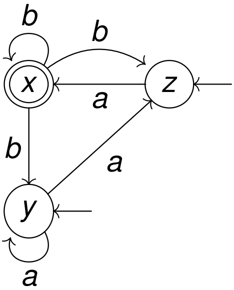
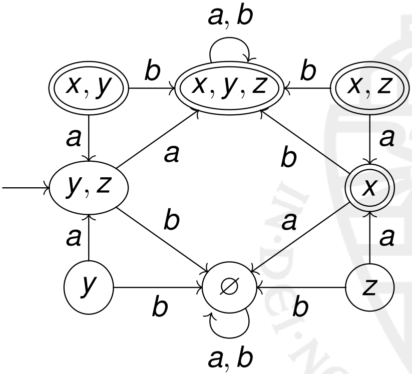




- new state space: $2^X = \{ V \mid V \subseteq \{x, y, z\} \}$

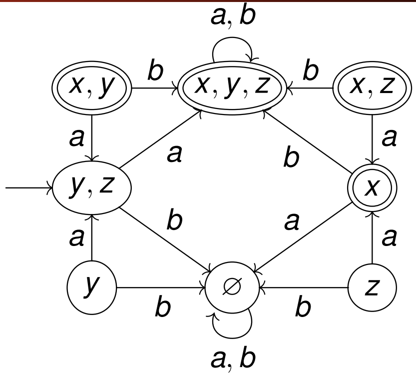
Making it deterministic again: *det*(*rev*(*X*))



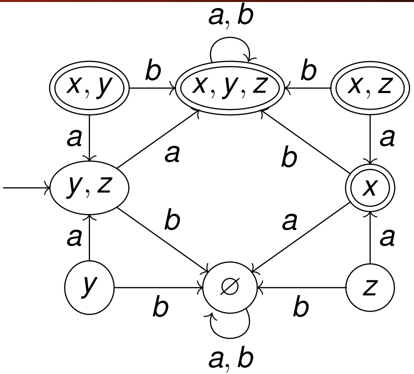


- new state space: $2^X = \{ V \mid V \subseteq \{x, y, z\} \}$
- $V \xrightarrow{a} W \qquad W = \{ w \mid \ v \xrightarrow{a} w \ , v \in V \}$

The automaton $det(rev(X))$ . . .



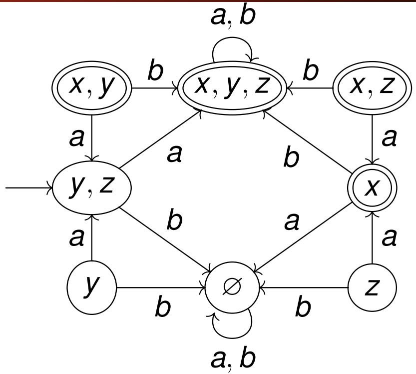
The automaton *det*(*rev*(*X*)) . . .



- . . . accepts the reverse of the language accepted by X:

$$L(\,det(rev(X))\,) \;=\; a\,\{a,b\}^* \;=\; reverse(\,L(X)\,)$$

The automaton *det*(*rev*(*X*)) . . .



- . . . accepts the reverse of the language accepted by *X*:

$$L(\,det(rev(X))\,) \;=\; a\{a,b\}^* \;=\; reverse(\,L(X)\,)$$

- . . . and is observable!

Today's Theorem

If: a deterministic automaton X is reachable and accepts $L(X)$

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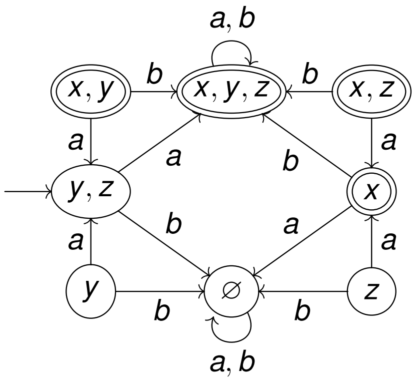
If: a deterministic automaton X is reachable and accepts $L(X)$

then: $det(rev(X))$ is minimal and

$$L(\, det(rev(X))\,) \;=\; reverse(\, L(X)\,)$$

Taking the reachable part of *det*(*rev*(*X*))



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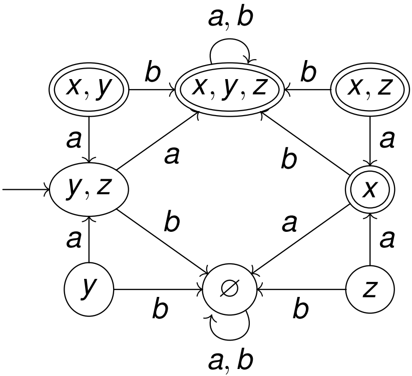
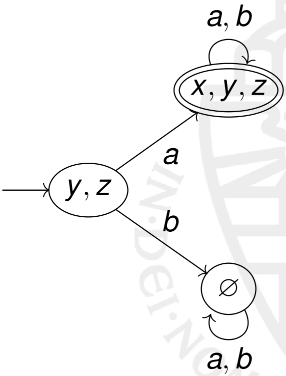
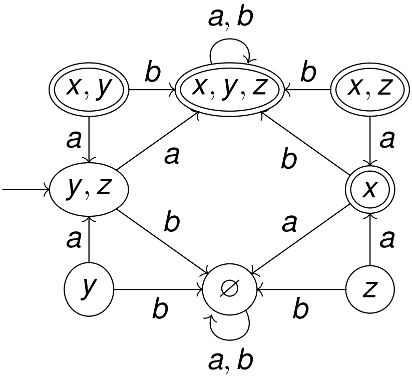
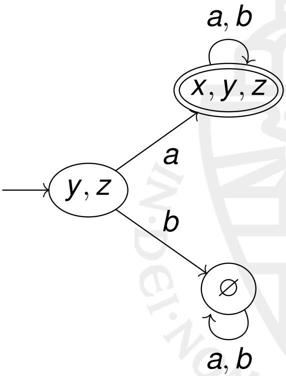




- $reach(det(rev(X)))$

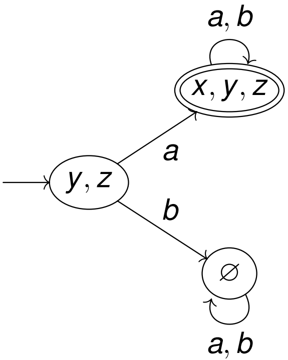
Taking the reachable part of *det*(*rev*(*X*))





- $reach(det(rev(X)))$ is reachable (by construction)

Repeating everything, now for *reach*(*det*(*rev*(*X*)))



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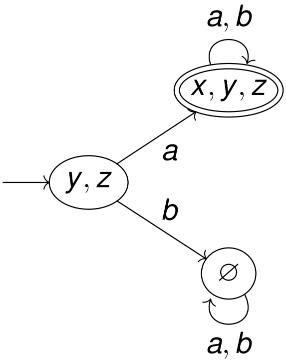

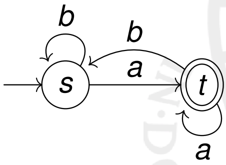

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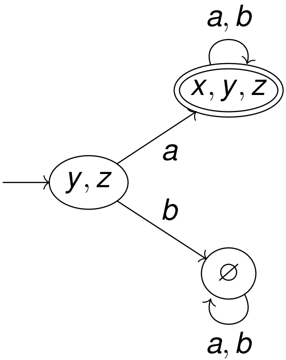
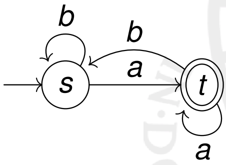
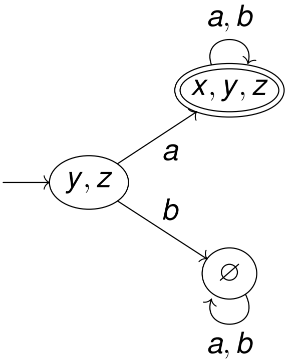
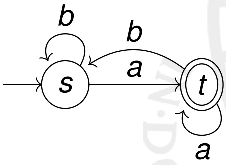




- . . . gives us $reach(det(rev(reach(det(rev(X))))))$

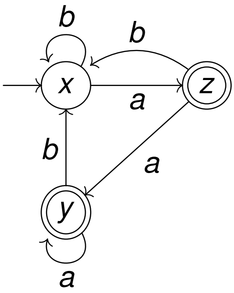
Repeating everything, now for *reach*(*det*(*rev*(*X*)))





- . . . gives us *reach*(*det*(*rev*(*reach*(*det*(*rev*(*X*))))))
- which is (reachable and) minimal and accepts $\{a, b\}^*\, a$.

All in all: Brzozowski's algorithm



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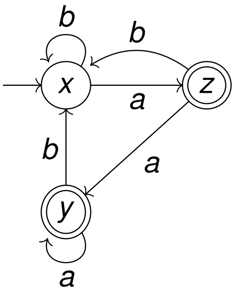
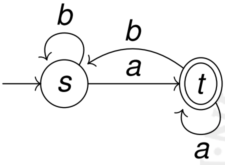
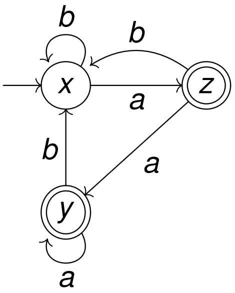
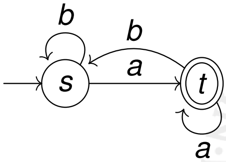




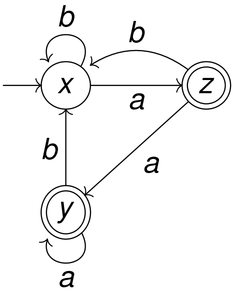
All in all: Brzozowski's algorithm

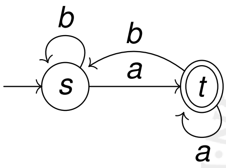




- X is reachable and accepts $\{a, b\}^* \, a$

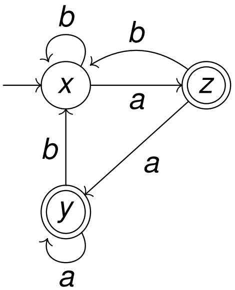
All in all: Brzozowski's algorithm

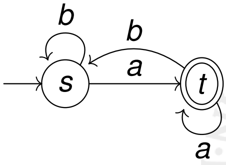




- X is reachable and accepts $\{a, b\}^* \, a$
- $reach(det(rev(reach(det(rev(X))))))$ also accepts $\{a, b\}^* \, a$

All in all: Brzozowski's algorithm





- X is reachable and accepts $\{a, b\}^*\, a$
- $reach(det(rev(reach(det(rev(X))))))$ also accepts $\{a, b\}^*\, a$
- ... and is minimal!!

Goal of the day

- Correctness of Brzozowski's algorithm (co)algebraically
- Generalizations to other types of automata

(Co)algebra

algebras:
$$\begin{tikzcd} F(X) \arrow[d, "f"'] \\ X \end{tikzcd}$$

coalgebras:
$$\begin{tikzcd} X \arrow[d, "f"'] \\ F(X) \end{tikzcd}$$

Examples of algebras

$$\begin{tikzcd} \mathbb{N} \times \mathbb{N} \arrow[d, "+"'] \\ \mathbb{N} \end{tikzcd}$$

Examples of algebras

$$
\begin{tikzcd}
\mathbb{N} \times \mathbb{N} \arrow[d, "+"'] \\
\mathbb{N}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
1 + \mathbb{N} \arrow[d, "{[0, S]}"'] \\
\mathbb{N}
\end{tikzcd}
\equiv
\begin{tikzcd}
1 \arrow[dr, "0"'] & & \mathbb{N} \arrow[dl, "S"] \\
& \mathbb{N} &
\end{tikzcd}
\equiv
\begin{tikzcd}
1 \arrow[dr, "0"'] & \\
& \mathbb{N} \arrow[d, "S"] \\
& \mathbb{N}
\end{tikzcd}
$$

Examples of coalgebras

$$\begin{tikzcd} X \arrow[d, "t"'] \\ \mathcal{P}(A \times X) \end{tikzcd} \qquad x \xrightarrow{a} y \quad \leftrightarrow \quad \langle a, y \rangle \in t(x)$$

Examples of coalgebras

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$$\begin{tikzcd} X \arrow[d, "\langle Left, label, Right \rangle"'] \\ X \times A \times X \end{tikzcd}$$

Examples of coalgebras

$$
\begin{tikzcd}
2^\omega \arrow[d, "\langle \text{head}, \text{tail} \rangle"'] \\
2 \times 2^\omega
\end{tikzcd}
\equiv
\begin{tikzcd}
& 2^\omega \arrow[dl, "\text{head}"'] \arrow[dr, "\text{tail}"] & \\
2 & & 2^\omega
\end{tikzcd}
\equiv
\begin{tikzcd}
& 2 \\
2^\omega \arrow[ur, "\text{head}"'] \arrow[d, "\text{tail}"] & \\
2^\omega &
\end{tikzcd}
$$

$$\text{head}((b_0, b_1, b_2, \ldots)) = b_0$$

$$\text{tail}((b_0, b_1, b_2, \ldots)) = (b_1, b_2, b_3 \ldots)$$

Homomorphisms

$$\begin{tikzcd} F(X) \arrow[r, "F(h)"] \arrow[d, "f"'] & F(Y) \arrow[d, "g"] \\ X \arrow[r, "h"'] & Y \end{tikzcd}$$

Homomorphisms

$$\begin{tikzcd} F(X) \arrow[r, "F(h)"] & F(Y) \\ X \arrow[u, leftarrow, "f"'] \arrow[r, "h"'] & Y \arrow[u, leftarrow, "g"'] \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ F(X) \arrow[r, "F(h)"'] & F(Y) \end{tikzcd}$$

Initiality, finality

$$
\begin{tikzcd}
F(A) \arrow[r, dashed, "F(h)"] \arrow[d, "\alpha"'] & F(X) \arrow[d, "f"] \\
A \arrow[r, dashed, "\exists !\, h"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, dashed, "\exists !\, h"] \arrow[d, "f"'] & Z \arrow[d, "\beta"] \\
F(X) \arrow[r, dashed, "F(h)"'] & F(Z)
\end{tikzcd}
$$

Initiality, finality

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F(A) \arrow[r, dashed, "F(h)"] \arrow[d, "\alpha"'] & F(X) \arrow[d, "f"] \\
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- initial algebras $\leftrightarrow$ induction

Initiality, finality

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- initial algebras $\leftrightarrow$ induction
- final coalgebras $\leftrightarrow$ coinduction

Automata, (co)algebraically

- Automata are complicated structures:

  part of them is algebra - part of them is coalgebra

Automata, (co)algebraically

- Automata are complicated structures:

  part of them is algebra - part of them is coalgebra
- (. . . in two different ways . . .)

A deterministic automaton

$$
\begin{tikzcd}
1 \arrow[dr, "i"] & & 2 \\
& X \arrow[ur, "f"] \arrow[d, "t"] & \\
& X^A &
\end{tikzcd}
$$

A deterministic automaton

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\begin{tikzcd}
1 \arrow[dr, "i"] & & 2 \\
& X \arrow[ur, "f"] \arrow[d, "t"] & \\
& X^A &
\end{tikzcd}
$$

where

$$1 = \{0\} \qquad 2 = \{0,1\} \qquad X^A = \{\, g \mid g : A \to X \,\}$$

$$x \xrightarrow{a} y \quad \leftrightarrow \quad t(x)(a) = y$$

$i(0) \in X$ is the initial state

x is final (or accepting) $\leftrightarrow$ $f(x) = 1$

Automata: algebra or coalgebra?

- initial state: algebraic – final states: coalgebraic

$$
\begin{tikzcd}
1 \arrow[dr, "i"] & & 2 \\
& X \arrow[ur, "f"] &
\end{tikzcd}
$$

Automata: algebra or coalgebra?

- initial state: algebraic – final states: coalgebraic

$$\begin{tikzcd} 1 \arrow[dr, "i"] & & 2 \\ & X \arrow[ur, "f"'] & \end{tikzcd}$$

- transition function: both algebraic and coalgebraic

$$\frac{X \xrightarrow{t} X^A}{\dfrac{X \longrightarrow (A \longrightarrow X)}{X \times A \xrightarrow{t} X}}$$

Automata: algebra and coalgebra!

$$
\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & & 2 \\
A^* \arrow[r, dashed, "r"'] \arrow[d, "\alpha"'] & X \arrow[r, dashed, "o"'] \arrow[ur, "f"] \arrow[d, "t"] & 2^{A^*} \arrow[u, "\epsilon?"'] \arrow[d, "\beta"] \\
(A^*)^A \arrow[r, dashed, "r^A"'] & X^A \arrow[r, dashed, "o^A"'] & (2^{A^*})^A
\end{tikzcd}
$$

Automata: algebra and coalgebra!

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To take home: this picture!! . . .

Automata: algebra and coalgebra!

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\end{tikzcd}
$$

To take home: this picture!! . . . which we'll explain next . . .

The "automaton" of languages

$$
\begin{tikzcd}
2 \\
2^{A^*} \arrow[u, "\epsilon?"'] \arrow[d, "\beta"] \\
(2^{A^*})^A
\end{tikzcd}
$$

$$\epsilon?(L) = 1 \leftrightarrow \epsilon \in L$$

$$2^{A^*} = \{g \mid g : A^* \to 2\} \cong \{L \mid L \subseteq A^*\}$$

$$\beta(L)(a) = L_a = \{w \in A^* \mid a \cdot w \in L\}$$

The "automaton" of languages

$$\begin{tikzcd} 2 \\ 2^{A^*} \arrow[u, "\epsilon?"'] \arrow[d, "\beta"'] \\ (2^{A^*})^A \end{tikzcd}$$

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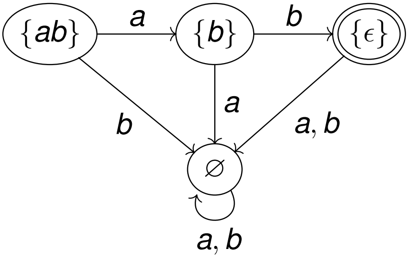
$$\beta(L)(a) = L_a = \{w \in A^* \mid a \cdot w \in L\}$$

- We say "automaton": it does not have an initial state.

The automaton of languages

- transitions: $L \xrightarrow{a} L_a$ where $L_a = \{w \in A^* \mid a \cdot w \in L\}$
- for instance:



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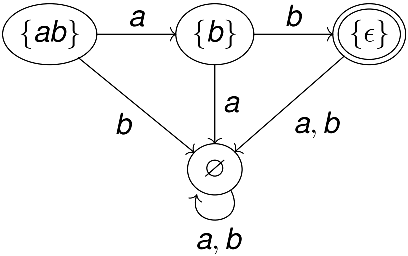

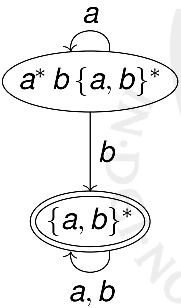

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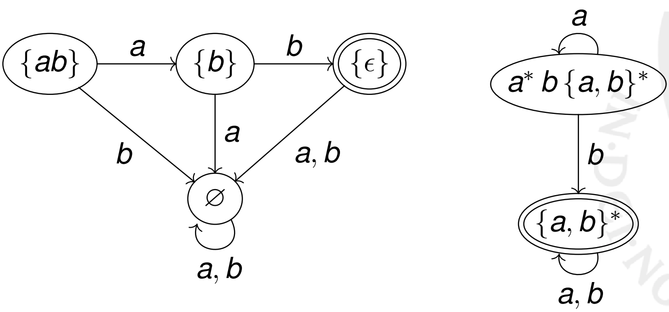

- note: every state L accepts . . .

The automaton of languages

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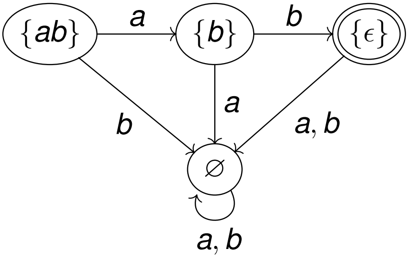

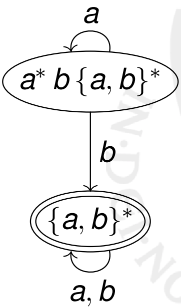

- note: every state L accepts . . . . . . the language L !!

The automaton of languages is . . . final

$$\begin{tikzcd}
& 2 \\
X \arrow[ur, "f"] \arrow[r, dashed, "\exists!\, o"'] \arrow[d, "t"] & 2^{A^*} \arrow[u, "\epsilon?"'] \arrow[d, "\beta"] \\
X^A \arrow[r, dashed, "o^A"'] & (2^{A^*})^A
\end{tikzcd}$$

$$\begin{aligned} o(x) &= \{w \in A^* \mid f(x_w) = 1\} \\ &= \text{the language accepted by } x \end{aligned}$$

The automaton of languages is . . . final

$$\begin{tikzcd} & 2 \\ X \arrow[ur, "f"] \arrow[r, dashed, "\exists!\, o"'] \arrow[d, "t"] & 2^{A^*} \arrow[u, "\epsilon?"'] \arrow[d, "\beta"] \\ X^A \arrow[r, dashed, "o^A"'] & (2^{A^*})^A \end{tikzcd}$$

$$o(x) = \{w \in A^* \mid f(x_w) = 1\}$$

$$= \text{the language accepted by } x$$

where: x_w is the state reached after inputting the word w,

and: $o^A(g) = o \circ g$, all $g \in X^A$.

Back to today's picture

$$
\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & & 2 \\
A^* \arrow[r, dashed, "r"'] \arrow[d, "\alpha"'] & X \arrow[r, dashed, "o"'] \arrow[ur, "f"] \arrow[d, "t"] & 2^{A^*} \arrow[u, "\epsilon?"'] \arrow[d, "\beta"] \\
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Back to today's picture

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\end{tikzcd}$$

On the right: final coalgebra

Back to today's picture

$$
\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & & 2 \\
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\end{tikzcd}
$$

On the right: final coalgebra

On the left: initial algebra . . .

The "automaton" of words

$$\begin{array}{c} 1 \\ \downarrow{\scriptstyle \epsilon} \\ A^* \\ \downarrow{\scriptstyle \alpha} \\ (A^*)^A \end{array}$$

ϵ is initial state

$$\alpha(w)(a) = w \cdot a$$

that is, transitions: $w \xrightarrow{a} w \cdot a$

The automaton of words is . . . initial

$$\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & \\
A^* \arrow[r, dashed, "\exists!\, r"'] \arrow[d, "\alpha"'] & X \arrow[d, "t"] \\
(A^*)^A \arrow[r, dashed, "r^A"'] & X^A
\end{tikzcd}$$

$$\begin{aligned}
i \in X &= \text{initial state} \\
&\ \text{(to be precise: } i(0)\text{)} \\
r(w) &= i_w \\
&= \text{the state reached from } i \\
&\ \text{after inputting } w
\end{aligned}$$

- Proof: easy exercise.
- Proof: formally, because A^* is an initial $1 + A \times (-)$-algebra!

Duality

- Reachability and observability are dual:

  Arbib and Manes, 1975.

- (here observable = minimal)

Reachability and observability

$$\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & & 2 \\
A^* \arrow[r, dashed, "r"'] \arrow[d, "\alpha"'] & X \arrow[ur, "f"] \arrow[r, dashed, "o"'] \arrow[d, "t"] & 2^{A^*} \arrow[u, "\epsilon?"'] \arrow[d, "\beta"] \\
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Reachability and observability

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(A^*)^A \arrow[r, dashed, "r^A"'] & X^A \arrow[r, dashed, "o^A"'] & (2^{A^*})^A
\end{tikzcd}$$

$$\begin{aligned}
r(w) &= \text{state reached on input } w \\
o(x) &= \text{language accepted by } x
\end{aligned}$$

Reachability and observability

$$
\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & & 2 \\
A^* \arrow[r, dashed, "r"'] \arrow[d, "\alpha"'] & X \arrow[r, dashed, "o"'] \arrow[ur, "f"] \arrow[d, "t"] & 2^{A^*} \arrow[u, "\epsilon?"'] \arrow[d, "\beta"] \\
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$$

$$
\begin{aligned}
r(w) &= \text{state reached on input } w \\
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- We call X reachable if r is surjective.

Reachability and observability

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(A^*)^A \arrow[r, dashed, "r^A"'] & X^A \arrow[r, dashed, "o^A"'] & (2^{A^*})^A
\end{tikzcd}$$

$$r(w) = \text{state reached on input } w$$

$$o(x) = \text{language accepted by } x$$

- We call X reachable if r is surjective.
- We call X observable (= minimal) if o is injective.

Reversing the automaton

- Reachability $\leftrightarrow$ observability
- Being precise about homomorphisms is crucial.
- Forms the basis for proof Brzozowski's algorithm.

Powerset construction

$$2^{(-)} \; : \quad \begin{array}{c} V \\ \Big\downarrow{\scriptstyle g} \\ W \end{array} \quad \mapsto \quad \begin{array}{c} 2^V \\ \Big\uparrow{\scriptstyle 2^g} \\ 2^W \end{array}$$

Powerset construction

$$2^{(-)} : \quad \begin{tikzcd} V \arrow[d, "g"'] \\ W \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} 2^V \\ 2^W \arrow[u, "2^g"'] \end{tikzcd}$$

where $2^V = \{S \mid S \subseteq V\}$ and, for all $S \subseteq W$,

$$2^g(S) = g^{-1}(S) \quad (= \{v \in V \mid g(v) \in S\})$$

Powerset construction

$$2^{(-)} : \quad \begin{array}{c} V \\ g \big\downarrow \\ W \end{array} \quad \mapsto \quad \begin{array}{c} 2^V \\ \big\uparrow 2^g \\ 2^W \end{array}$$

where $2^V = \{S \mid S \subseteq V\}$ and, for all $S \subseteq W$,

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- This construction is contravariant !!

Powerset construction

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- This construction is contravariant !!
- Note: if g is surjective, then 2^g is injective.

Reversing transitions

$$
\begin{tikzcd}
X \arrow[d, "t"'] \\
X^A
\end{tikzcd}
$$

Reversing transitions

$$
\begin{array}{c||c}
\begin{tikzcd} X \arrow[d, "t"'] \\ X^A \end{tikzcd} & \begin{tikzcd} X \times A \arrow[d] \\ X \end{tikzcd}
\end{array}
$$

Reversing transitions

$$
\begin{array}{c|c}
\begin{tikzcd} X \arrow[d, "t"'] \\ X^A \end{tikzcd} & \begin{tikzcd} X \times A \arrow[d] \\ X \end{tikzcd}
\end{array}
\quad \xmapsto{\;2^{(-)}\;} \quad
\begin{tikzcd} 2^{X \times A} \\ 2^X \arrow[u] \end{tikzcd}
$$

Reversing transitions

$$
\begin{array}{c|c}
X & X \times A \\
t \downarrow & \downarrow \\
X^A & X
\end{array}
\quad \overset{2^{(-)}}{\longmapsto} \quad
\begin{array}{c|c}
2^{X \times A} & (2^X)^A \\
\uparrow & \uparrow \\
2^X & 2^X
\end{array}
$$

Reversing transitions

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\begin{array}{c|c}
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\Big\downarrow t & \Big\downarrow \\
X^A & X
\end{array}
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\Big\uparrow & \Big\uparrow & \Big\downarrow 2^t \\
2^X & 2^X & (2^X)^A
\end{array}
$$

Initial $\leftrightarrow$ final

$$\begin{tikzcd} 1 \arrow[dr, "i"] & \\ & X \end{tikzcd}$$

Initial $\leftrightarrow$ final

$$
\begin{tikzcd}
1 \arrow[dr, "i"] & & & & & 2 \\
& X & \arrow[r, mapsto, "2^{(-)}"] & {} & 2^X \arrow[ur, "2^i"] &
\end{tikzcd}
$$

Initial $\leftrightarrow$ final

$$1 \xrightarrow{i} X \qquad \xmapsto{2^{(-)}} \qquad 2^X \xrightarrow{2^i} 2$$

$$X \xrightarrow{f} 2$$

Initial $\leftrightarrow$ final

$$\left(1 \xrightarrow{i} X \right) \quad \xmapsto{2^{(-)}} \quad \left(2^X \xrightarrow{2^i} 2 \right)$$

$$\left(X \xrightarrow{f} 2 \right) \quad \longleftrightarrow \quad \left(1 \xrightarrow{f} 2^X \right)$$

Reversing the entire automaton

$$\begin{tikzcd} 1 \arrow[dr, "i"] & & 2 \\ & X \arrow[ur, "f"] \arrow[d, "t"] & \\ & X^A & \end{tikzcd}$$

Reversing the entire automaton

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\end{tikzcd}
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- Initial and final are exchanged . . .

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- Initial and final are exchanged . . .
- transitions are reversed . . .

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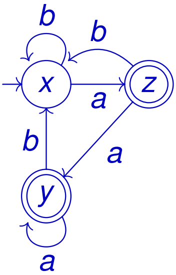
1 \arrow[dr, "f"] & & 2 \\
& 2^X \arrow[ur, "2^i"] \arrow[d, "2^t"] & \\
& (2^X)^A &
\end{tikzcd}
$$

- Initial and final are exchanged . . .
- transitions are reversed . . .
- and the result is again deterministic!

Our previous example

$X =$



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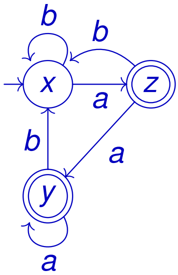
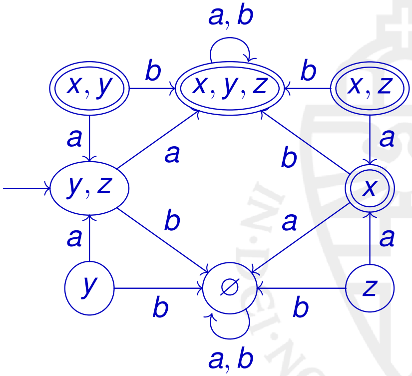
$X =$ 

$2^X =$ 

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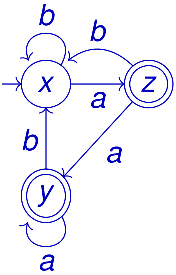
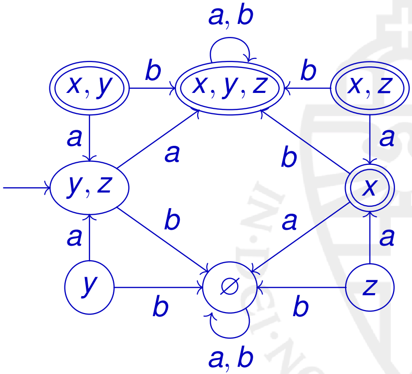


$2^X =$



- Note that X has been reversed and determinized:

$$2^X = det(rev(X))$$

Proving today's Theorem

If: a deterministic automaton X is reachable and accepts $L(X)$

Proving today's Theorem

If: a deterministic automaton X is reachable and accepts $L(X)$

then: 2^X ($=$ $det(rev(X))$) is minimal/observable and

$$L(2^X) = reverse(L(X))$$

Proof: by reversing $A^* \xrightarrow{r} X$

$$
\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & \\
A^* \arrow[r, "r"'] \arrow[d, "\alpha"'] & X \arrow[d, "t"] \\
(A^*)^A \arrow[r] & X^A
\end{tikzcd}
$$

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A^* \arrow[r, "r"'] \arrow[d, "\alpha"'] & X \arrow[d, "t"] & \arrow[r, mapsto, "2^{(-)}"] & 2^X \arrow[ur, "2^i"] \arrow[r, "2^r"'] \arrow[d, "2^t"'] & 2^{A^*} \arrow[u, "2^\epsilon"'] \arrow[d, "2^\alpha"] \\
(A^*)^A \arrow[r] & X^A & & (2^X)^A \arrow[r] & (2^{A^*})^A
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 & 2 \\
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- X becomes 2^X

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- X becomes 2^X
- initial automaton A^* becomes (almost) final automaton 2^{A^*}

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$$

- X becomes 2^X
- initial automaton A^* becomes (almost) final automaton 2^{A^*}
- r is surjective $\Rightarrow$ 2^r is injective

Reachable becomes observable

$$\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & \\
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(2^X)^A \arrow[r] & (2^{A^*})^A \arrow[r, dashed] & (2^{A^*})^A
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- If r is surjective then (2^r and hence) $rev \circ 2^r$ is injective.

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- If r is surjective then (2^r and hence) $rev \circ 2^r$ is injective.
- That is, 2^X is observable (= minimal).

Summarizing

$$\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & & 2 \\
A^* \arrow[r, dashed, "r"'] \arrow[d, "\alpha"'] & X \arrow[ur, "f"] \arrow[d, "t"] & \\
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- If: X is reachable, i.e., r is surjective

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- If: X is reachable, i.e., r is surjective

  then: $rev \circ 2^r$ is injective, i.e., 2^X is observable = minimal.
- And: $rev(2^r(f)) = rev(o(i))$, i.e., $L(\,2^X\,) \;=\; reverse(\,L(X)\,)$

Corollary: Brzozowski's algorithm

- X becomes 2^X, accepting $\mathit{reverse}(L(X))$

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- X becomes 2^X, accepting *reverse*$(L(X))$
- take reachable part: $Y = \mathit{reachable}(2^X)$
- Y becomes 2^Y, which is minimal and accepts

$$\mathit{reverse}(\mathit{reverse}(L(X))) = L(X)$$

Generalizations

$$
\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & & 2 \\
A^* \arrow[r, dashed, "r"'] \arrow[d, "\alpha"'] & X \arrow[r, dashed, "o"'] \arrow[d, "t"] \arrow[ur, "f"] & 2^{A^*} \arrow[u, "\epsilon?"'] \arrow[d, "\zeta"] \\
(A^*)^A \arrow[r, dashed, "r^A"'] & X^A \arrow[r, dashed, "o^A"'] & (2^{A^*})^A
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[d, "\epsilon"'] \arrow[dr, "i"] & & B \\
A^* \arrow[r, dashed, "r"'] \arrow[d, "\alpha"'] & X \arrow[r, dashed, "o"'] \arrow[d, "t"] \arrow[ur, "f"] & B^{A^*} \arrow[u] \arrow[d] \\
(A^*)^A \arrow[r, dashed, "r^A"'] & X^A \arrow[r, dashed, "o^A"'] & (B^{A^*})^A
\end{tikzcd}
$$

- A Brzozowski minimization algorithm for Moore automata.

$$B^X = \{\varphi \mid \varphi \colon X \to B\} \qquad B^f(\varphi) = \varphi \circ f$$

Further generalizations

- Moore automata generalization: uniform algorithm for decorated traces and must testing (joint work with Bonchi, Caltais and Pous);
- Further generalizations to non-deterministic and weighted automata.

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Coalgebra is not only semantics but also algorithms!

CoCaml: Programming with Coinductive Types

(joint work with Jean-Baptiste Jeannin and Dexter Kozen)

Computing with Coalgebraic Data

- Inductive datatypes and functions on those are well-understood; coinductive datatypes often considered difficult to handle, not many programming languages offer the constructs for them.
- OCaml offers the possibility of defining coinductive datatypes, but the means to define recursive functions on them are limited.
- Often the obvious definitions do not halt or provide the wrong solution.
- Even so, there are often perfectly good solutions (examples forthcoming!)
- We show how to extend the language to allow it!

Motivating example

```
type list = N | C of int * list

let rec ones = C(1, ones);; 1,1,1,1,...
let rec alt = C(1, C(2, alt));; 1,2,1,2,...
```

Motivating example

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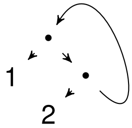
```

Infinite lists but… regular:



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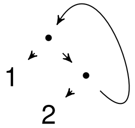


A simple function:

```
let set l = match l with
| N -> N
| C(h, t) -> (insert h (set t));;
```

We expect `set ones = {1}` and `set alt = {1,2}`.

What is the problem?

- The function definition above will not halt in OCaml…
- even though it is clear what the answer should be;

What is the problem?

- The function definition above will not halt in OCaml...
- even though it is clear what the answer should be;
- Note that this is not a corecursive definition: we are not asking for a greatest solution or a unique solution in a final coalgebra,
- but rather a least solution in a different ordered domain from the one provided by the standard semantics of recursive functions.
- Standard semantics: least solution in the flat Scott domain with bottom element $\perp$ representing nontermination
- Intended semantics: least solution in a different CPO, namely $(\mathcal{P}(\mathbb{Z}), \subseteq)$ with bottom element $\varnothing$.

Motivating example c'd

We would like to use (almost) the same definition and get the intended solution. . .

```
let set l = match l with
| N -> N
| C(h, t) -> (insert h (set t));;
```

Motivating example c'd

We would like to use (almost) the same definition and get the intended solution...

```
let set l = match l with
| N -> N
| C(h, t) -> (insert h (set t));;
```

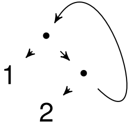
We change it to:

```
let corec[iterator(N)] set l = match l with
| N -> N
| C(h, t) -> insert h (set t);;
```

The construct `corec` with the parameter `iterator(N)` specifies to the compiler how to solve equations.

Motivating example c'd

For instance, for the infinite list `alt`:



the compiler will generate two equations:

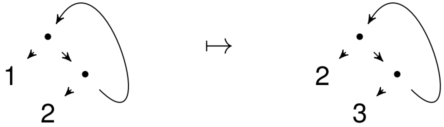
```
set(x) = insert 1 (set(y))
set(y) = insert 2 (set(x))
```

then solve them using `iterator` (least fixed point) which will produce the intended set $\{1, 2\}$.

Motivating example c'd

```
let map f = match arg with
| N -> N
| C(h, t) -> C(f(h), map(f,t));;
```

We would like: `map plusOne alt` to produce the infinite list $2, 3, 2, 3, \ldots$:



This is not a least fixed point computation anymore but rather a solution in the final coalgebra.

Another Example

Free variables of a λ-term

```
type term =
   | Var of string
   | App of term *
term
   | Lam of string *
term
```

x

$(f \ \ e)$

$\lambda x.e$

```
let rec fv = function
   | Var v -> {v}
   | App(t1,t2) -> fv t1 U fv t2
   | Lam(x,t) -> (fv t) - {x}
```

Another Example

But what about infinitary λ-terms (λ-coterms)?

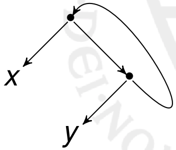
```
type term =
   | Var of string
   | App of term *
term
   | Lam of string *
term
```

x

$(f \;\; e)$

$\lambda x.e$

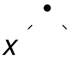
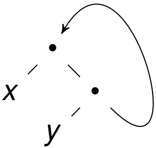
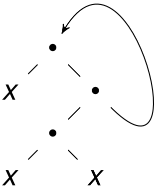
```
let rec fv = function
   | Var v -> {v}
   | App(t1,t2) -> fv t1 U fv t2
   | Lam(x,t) -> (fv t) - {x}
```



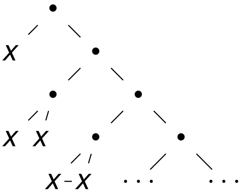

```
let rec t = App(Var "x", App(Var "y", t))
```

We would like: `fv t = {x,y}` (again LFP).

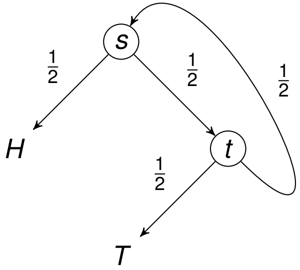
Substitution

Replace *y* by  in  to get  .

The usual semantics would infinitely unfold the term on the left, generating instead:

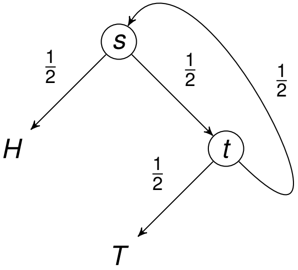


Probabilistic Protocols



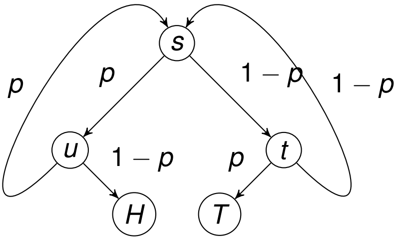
$$\mathrm{Pr}_H(s) = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \cdots = \frac{2}{3}$$
$$\mathrm{Pr}_H(t) = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots = \frac{1}{3}$$

Probabilistic Protocols



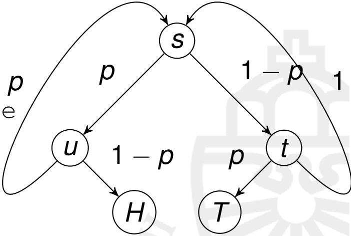
$$\mathrm{Pr}_H(s) = \frac{1}{2} + \frac{1}{2} \cdot \mathrm{Pr}_H(t)$$
$$\mathrm{Pr}_H(t) = \frac{1}{2} \cdot \mathrm{Pr}_H(s)$$

The Von Neumann Trick



$$\begin{aligned}
\Pr_H(s) &= p \cdot \Pr_H(u) + (1-p) \cdot \Pr_H(t) \\
\Pr_H(u) &= (1-p) + p \cdot \Pr_H(s) \\
\Pr_H(t) &= (1-p) \cdot \Pr_H(s)
\end{aligned}$$

The Von Neumann Trick




```
type state =
  | H
  | T
  | Flip of float * state * state

let rec pr_heads s = function
  | H -> 1.
  | T -> 0.
  | Flip(p,u,v) ->
     p *. (pr_heads u) +. (1 -. p) *. (pr_heads v)

let rec s = Flip(.345,u,t)
and u = Flip(.345,H,s)
and t = Flip(.345,T,s)

print p_heads s
```

Theoretical Foundations

- Well-founded coalgebras [Taylor 99]
- Recursive coalgebras [Adámek, Lücke, Milius 07]
- Elgot algebras [Adámek, Milius, Velebil 06]
- Corecursive algebras [Capretta, Uustalu, Vene 09]

Ingredients:

- Functor F (usually polynomial or power set)
- domain: an F-coalgebra (C, γ)
- range: an F-algebra (A, α)

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "\gamma"'] & A \\ FC \arrow[r, "Fh"'] & FA \arrow[u, "\alpha"'] \end{tikzcd}$$

What about Non-Well-Founded Coalgebras?

The foundations existing so far were for unique solutions; we want alternative solutions.

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$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "\gamma"'] & A \\ FC \arrow[r, "Fh"'] & FA \arrow[u, "\alpha"'] \end{tikzcd}$$

- Even if (C, γ) is not well-founded, the diagram may still have a canonical solution, provided (A, α) comes equipped with a method for solving systems of equations
- The diagram specifies the system to be solved
- The variables are the elements of C and h is their interpretation in A

The general idea

The programmer specifies the equations as usual with an extra parameter, like in:

```
let corec[iterator(N)] set l = match l with
| N -> N
| C(h, t) -> insert h (set t);;
```

The general idea

The programmer specifies the equations as usual with an extra parameter, like in:

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let corec[iterator(N)] set l = match l with
| N -> N
| C(h, t) -> insert h (set t);;
```
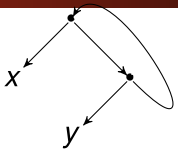

The compiler generates equations and solves them using the extra parameter.

Free Variables of a λ-Coterm

The free variables of



are $\{x, y\}$

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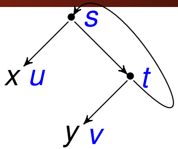


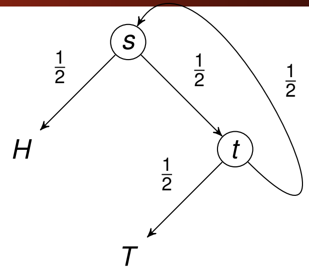
are $\{x, y\}$

$$
\begin{aligned}
fv(s) &= fv(u) \cup fv(t) \\
fv(t) &= fv(v) \cup fv(s) \\
fv(u) &= \{x\} \\
fv(v) &= \{y\}
\end{aligned}
$$

The least solution in $(\mathcal{P}(\mathrm{Var}), \subseteq)$ is $\{x, y\}$

Standard semantics: $A \cup \perp = \perp$, whereas here $A \cup \varnothing = A$

Example: Probabilistic Protocols



$$\mathrm{Pr}_H(s) = \tfrac{1}{2} + \tfrac{1}{2} \cdot \mathrm{Pr}_H(t) \qquad \mathrm{Pr}_H(t) = \tfrac{1}{2} \cdot \mathrm{Pr}_H(s)$$

- Can calculate expected running times, higher moments, outcome functions similarly
- These are all least solutions in an appropriate ordered domain—in the above example, $([0,1], \leq)$

Other Non-Well-Founded Examples

- static analysis, abstract interpretation
- *p*-adic arithmetic
- automata constructions

Implementation

- We implemented `corec` constructor which takes a solver as a parameter
- We implemented several general solvers: least fixed point, unique solution in a final coalgebra, gaussian elimination, …

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- We implemented `corec` constructor which takes a solver as a parameter
- We implemented several general solvers: least fixed point, unique solution in a final coalgebra, gaussian elimination, ...
- Solvers are implemented directly in the interpreter, as transformers from an abstract syntax tree to another abstract syntax tree.
- Future: to provide tools to manipulate the abstract syntax tree allowing programmers to easily specify their solver.

CoCaml

- CoCaml offers new program constructs and functionalities to implement functions on coinductive structures.
- Examples illustrate the need for new constructs
- New constructs enable allow definitions very much in the style of standard recursive functions.

`http://www.cs.cornell.edu/Projects/CoCaml/`

Thanks!