Convex Bisimilarity and Real-valued Modal Logics

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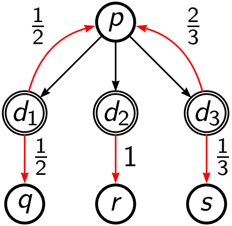
Probabilistic Nondeterministic Transition Systems (PNTS's)

- a.k.a, Probabilistic Automata, Markov Decision Processes, Simple Segala Systems



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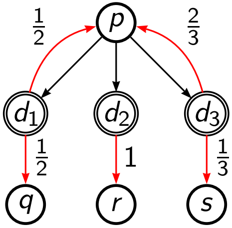

- F-coalgebras (X, α) of $F(X) = P(D(X))$.
  - $P(X) =$ powerset of X
  - $D(X) =$ discrete probability distributions on X

Logics for PNTS's

Can be organized in three categories:

1. PCTL, PCTL* and similar logics ($\sim$20years old)
   - Used in practice because can express useful properties.
   - Main tool is Model-Checking, no much else.
   - Logically induce non-standard notions of behavioral equivalence

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3. Quantitative (Real-valued) logics.

Quantitative Logics

Given a PNTS's (X, α)

- **Semantics**: $[\![\phi]\!] : X \to \mathbb{R}$
  - E.g., $[\![\phi \wedge \psi]\!](x) = \min\big([\![\phi]\!](x), [\![\psi]\!](x)\big)$
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  - Game Semantics: Two-Player Stochastic Games
- Under development: Model Checking algorithms, Compositional Proof Systems, ...

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- **Sound and Complete Axiomatizations**?
  - Proof Systems?

Behavioral Equivalences for PNTS's

Several have been proposed in the literature.

Coalgebra shed some light: **Cocongruence**

Definition Given F-coalgebra (X, α), the equivalence relation $E \subseteq X \times X$ is a cocongruence iff

$$(x, y) \in E \quad \Rightarrow \quad \big(\alpha(x), \alpha(y)\big) \in \hat{E}.$$

Examples: Coalgebra (X, α)

- of powerset functor P. Given $A, B \in P(X)$
  - $(A, B) \in \hat{E}_P \quad \Leftrightarrow \quad \big\{[x]_E \mid x \in A\big\} = \big\{[x]_E \mid x \in B\big\}$

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- of PD functor (PNTS's). Given $A, B \in PD(X)$
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Cocongruence for PNTS's was introduced (concretely) by Roberto Segala in his PhD thesis (1994).

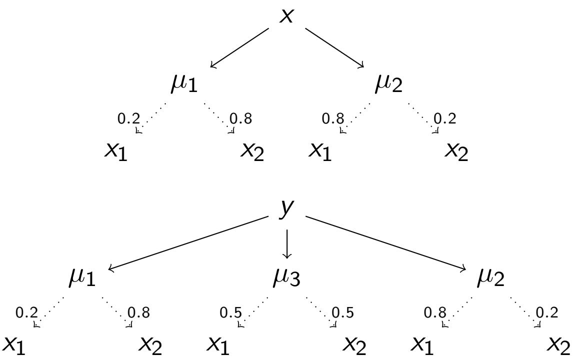
- Standard Bisimilarity for PNTS's.

Def: Given (X, α), an equivalence $E \subseteq X \times X$ is a standard bisimulation if

- for all $x \rightarrow \mu$ there exists $y \rightarrow \nu$ such that $(\mu, \nu) \in \hat{E}_D$, and
- for all $y \rightarrow \nu$ there exists $x \rightarrow \mu$ such that $(\mu, \nu) \in \hat{E}_D$,

where $x \rightarrow \mu$ means $\mu \in \alpha(x)$.

Two states (x, y) which are not standard bisimilar.



Under the assumption that x_1 and x_2 are distinguishable.

Convex Bisimilarity

Def: Given (X, α), an equivalence $E \subseteq X \times X$ is a convex bisimulation if

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Cocongruence of F-coalgebras for $F = P_c D$

- $P_c D \;=\;$ Convex Sets of Probability Distributions.

$$(X, \alpha : X \to PD(X)) \quad \xrightarrow{H} \quad (X, \alpha : X \to P_c D(X))$$

Standard Bisimilarity Convex Bisimilarity

Fact: Expressive logics for PNTS's can not distinguish convex bisimilar states.

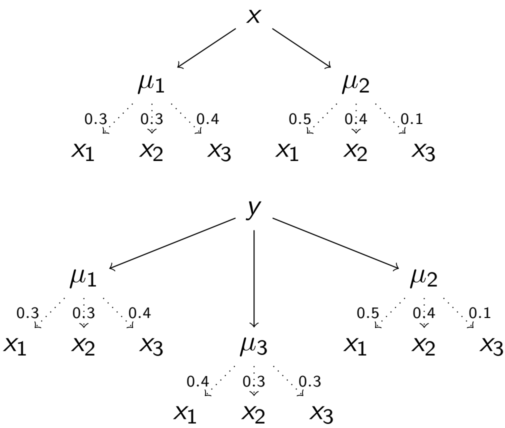
- PCTL, PCTL* and the $\mathbb{R}$-valued μ-Calculi

$$\text{convex bisim.} \subsetneq \text{PCTL}^* \subsetneq \text{PCTL}$$

$$\text{convex bisim.} \subseteq_? \text{quantitative } \mu\text{-calculi}$$

Natural question: does Convex Bisimilarity distinguish too much?

Example of (x, y) not Convex Bisimilar:



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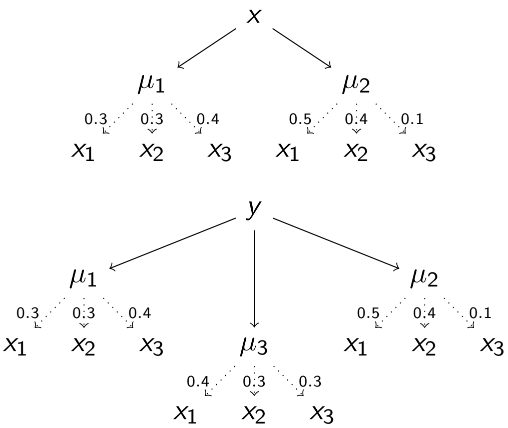

Suppose we want to observe *event* $\Phi = \{x_1\}$.

- y can exhibit Φ with probability $[0.3, 0.5]$. But also x can!

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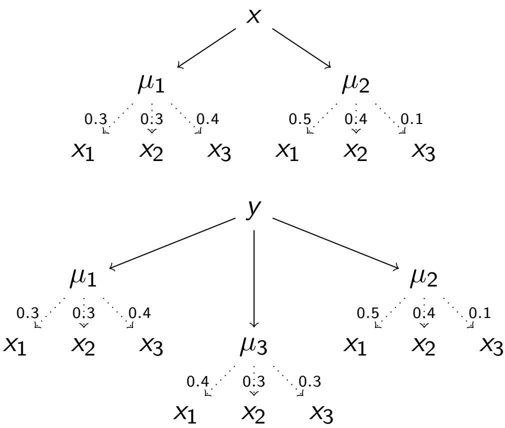

Suppose we want to observe *event* $\Phi = \{x_2\}$.

- y can exhibit Φ with probability $[0.3, 0.4]$. But also x can!

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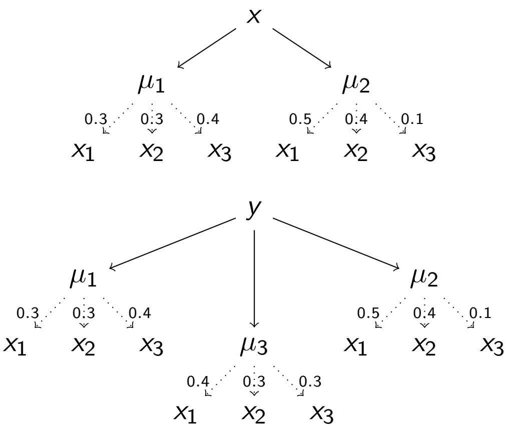

Suppose we want to observe *event* $\Phi = \{x_1, x_2\}$.

- y can exhibit Φ with probability $[0.6, 0.9]$. But also x can!

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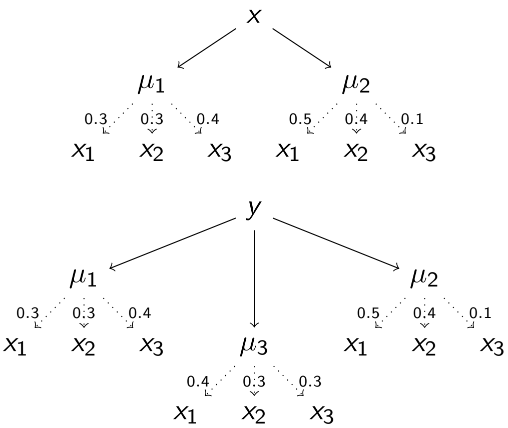

As a matter of fact, for all *events* $\Phi \subseteq \{x_1, x_2, x_3\}$.

- y can exhibit Φ with probability $[\lambda_1, \lambda_2]$ iff x can!

We considered events $\Phi \subseteq \{x_1, x_2, x_3\}$?

- What about Random Variables $f : \{x_1, x_2, x_3\} \to \mathbb{R}$?

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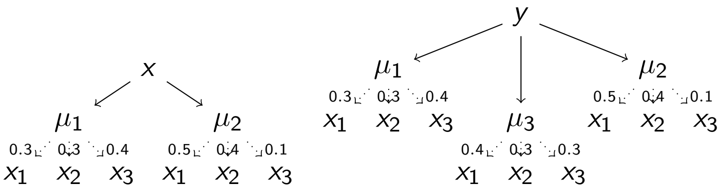
Example: $f(x_1) = 60$, $f(x_2) = 0$, $f(x_3) = 50$.



Expected values: $E_{\mu_1}(f) = 38$, $E_{\mu_2}(f) = 35$, $E_{\mu_3}(f) = 39$.

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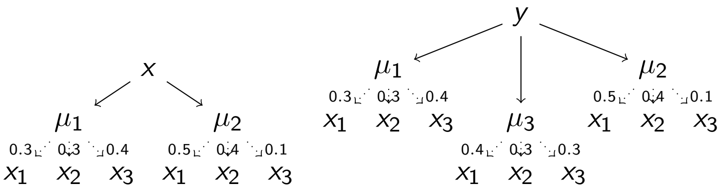

Expected values: $E_{\mu_1}(f) = 38$, $E_{\mu_2}(f) = 35$, $E_{\mu_3}(f) = 39$.

- The average resulting from interactions on y **CAN BE** greater than 38 (and always is smaller than 39)
- The average resulting from interactions on y **CAN NOT BE** greater than 38

Upper Expectation Bisimilarity

Upper Expectation Functional: Given a set A of probability distributions on X, define $ue_A : (X \to \mathbb{R}) \to \mathbb{R}$ as:

$$ue_A(f) = \sup\{E_\mu(f) \mid \mu \in A\}$$

Upper Expectation (UE) Bisimulation. Given a PNTS (X, α), an equivalence relation $E \subseteq X \times X$ is a UE-bisimulation if

- $ue_{\alpha(x)}(f) = ue_{\alpha(y)}(f)$

for all E-invariant $f : X \to \mathbb{R}$, i.e., such that if $(z, w) \in E$ then $f(z) = f(w)$.

Functional Analysis

Functionals of type $(X \to \mathbb{R}) \to \mathbb{R}$, e.g. $C(X)^*$, are well studied in Functional Analysis.

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Theorem: Let X be a finite set and $A \in PD(X)$ a set of probability distributions. Then:

- $ue_A = ue_{\overline{H}(A)}$
- $\{\mu \mid \forall f : X \to \mathbb{R}.(\mu(f) \leq ue_A(f))\} = \overline{H}(A)$

where $\overline{H}(A)$ is the *closed convex hull* of A.

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Message: $ue_A : (X \to \mathbb{R}) \to \mathbb{R}$ and $\overline{H}(A)$ are the same thing.

Consequence

UE-bisimilarity = cocongruence for $P_{cc}D$-coalgebras.

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Therefore we have:

- Strong reasons for equating UE-bisimilar states (prob. schedulers)
- Strong reasons for distinguishing not UE-bisimilar states ($\mathbb{R}$-valued experiments).

Back to Logic!

PNTS	$\big(X, \alpha : X \to P_{cc}D(X)\big)$	$x \mapsto A_x$
PNTS	$\big(X, \alpha : X \to (X \to \mathbb{R}) \to \mathbb{R}\big)$	$x \mapsto ue_A$
PNTS	$\big(X, \alpha : (X \to \mathbb{R}) \to (X \to \mathbb{R})\big)$	$f \mapsto \lambda x.(ue_{\alpha(x)}(f))$

Denote with $\Diamond_\alpha : (X \to \mathbb{R}) \to (X \to \mathbb{R})$ the latter presentation.

Given a PNTS (X,α), $\mathbb{R}$-valued Modal logics have semantics:

$$[\![\phi]\!] : X \to \mathbb{R}.$$

and, in particular (for all the logics in the literature)

$$[\![\Diamond\phi]\!] = \Diamond_\alpha([\![\phi]\!]) \quad \stackrel{\text{def}}{=} \sup\{E_\mu([\![\phi]\!]) \mid \mu \in \alpha(x)\}$$

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The several logics in the literature differ on the choice of other connectives:

- $\llbracket \underline{1} \rrbracket (x) = 1$,
- $\llbracket \phi \sqcap \psi \rrbracket (x) = \min\{\llbracket \phi \rrbracket (x), \llbracket \psi \rrbracket (x)\}$
- ...

Functional Analysis - Again

Let (X, α) be a PNTS. Then $\Diamond_\alpha : (X \to \mathbb{R}) \to (X \to \mathbb{R})$ satisfies:

1. (Monotone) if $f \sqsubseteq g$ then $\Diamond_\alpha(f) \sqsubseteq \Diamond_\alpha(f)$
2. (Sublinear) $\Diamond_\alpha(f + g) \sqsubseteq \Diamond_\alpha(f) + \Diamond_\alpha(g)$
3. (Positive Affine Homogeneous)
   $\Diamond(\lambda_1 f + \lambda_2 \underline{1}) = \lambda_1 \Diamond_\alpha(f) + \lambda_2 \Diamond_\alpha \underline{1}$, for all $\lambda_1 \geq 0$, $\lambda_2 \in \mathbb{R}$
4. $\Diamond_\alpha(\underline{1}) \in X \to \{0, 1\}$

Completeness: Furthermore, every $(X \to \mathbb{R}) \to (X \to \mathbb{R})$ with these properties is $F = \Diamond_\alpha$ for a unique PNTS (X, α).

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Riesz Logic: $\phi ::= \underline{1} \mid f + g \mid \lambda f \mid f \sqcup g \mid \Diamond\phi$.

- Semantics interpreted on (X, α):
  - $[\![\underline{1}]\!](x) = 1$,
  - $[\![\phi + \psi]\!](x) = [\![\phi]\!](x) + [\![\psi]\!](x)$
  - $[\![\Diamond\phi]\!] = \Diamond_\alpha([\![\phi]\!])$

Theorems: Given a PNTS (X, α)

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- Completeness: if x and y are not UE-bisimilar then there is some ϕ such that $[\![\phi]\!](x) \neq [\![\phi]\!](y)$.
- We have a sound and complete axiomatization
  - Axioms from unitary Riesz spaces, plus
  - Axioms for $\Diamond$.

This is a general framework!!!

Example 1: The class of PNTS's that beside

1. (Monotone) if $f \sqsubseteq g$ then $\Diamond_\alpha(f) \sqsubseteq \Diamond_\alpha(f)$
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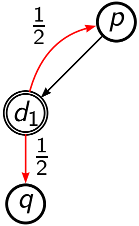
3. (Positive Affine Homogeneous) $\Diamond(\lambda_1 f + \lambda_2 \underline{1}) = \lambda_1 \Diamond_\alpha(f) + \lambda_2 \Diamond_\alpha \underline{1}$, for all $\lambda_1 \geq 0$, $\lambda_2 \in \mathbb{R}$
4. $\Diamond_\alpha(\underline{1}) \in X \to \{0,1\}$

also satisfy

- (Linearity) $\Diamond_\alpha(f+g) = \Diamond_\alpha(f) + \Diamond_\alpha(g)$

are **Markov processes**, i.e., PNTS (X, α) such that

- For all states $x \in X$, either $\alpha(x) = \{\mu\}$ or $\alpha(x) = \emptyset$


$\frac{1}{2}$
p
d_1
$\frac{1}{2}$
q

This is a general framework!!!

Example 2: The class of PNTS's that beside

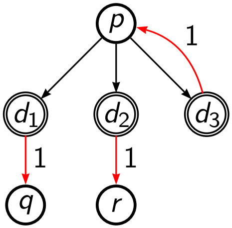
1. (Monotone) if $f \sqsubseteq g$ then $\Diamond_\alpha(f) \sqsubseteq \Diamond_\alpha(f)$
2. (Sublinear) $\Diamond_\alpha(f+g) \sqsubseteq \Diamond_\alpha(f) + \Diamond_\alpha(g)$
3. (Positive Affine Homogeneous)
   $\Diamond(\lambda_1 f + \lambda_2 \underline{1}) = \lambda_1 \Diamond_\alpha(f) + \lambda_2 \Diamond_\alpha \underline{1}$, for all $\lambda_1 \geq 0$, $\lambda_2 \in \mathbb{R}$
4. $\Diamond_\alpha(\underline{1}) \in X \rightarrow \{0,1\}$

also satisfy

- (Join preserving) $\Diamond(f \sqcup g) = \Diamond(f) \sqcup \Diamond(g)$.

are **Kripke frames**, i.e., PNTS (X, α) such that

- For all states $x \in X$ every $\mu \in \alpha(x)$ is a Dirac distribution.


p
1
d1
d2
d3
1
1
q
r

A quick note about μ-Calculi

The Łukasiewicz μ-Calculus (Łμ) is a $[0, 1]$-valued logic

- Introduced in my PhD thesis,
- (co)inductived fixed points (μ-Calculus)
- capable of encoding PCTL

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- The logic of MV-algebra.

We can apply a variant of the Yosida Representation Theorem:

- All MV-algebras are of the form $X \to [0, 1]$

Theorem: Łμ formulas are dense in $X \to [0, 1]$.

Summary

New prospective on Convex (closed) Bisimilarity

- in terms of UE-bisimilarity,
- motivated by $\mathbb{R}$-valued experiments $X \rightarrow \mathbb{R}$,
- concrete reason to distinguish between not UE-bisimilar states.

Summary

New prospective on Convex (closed) Bisimilarity

- in terms of UE-bisimilarity,
- motivated by $\mathbb{R}$-valued experiments $X \to \mathbb{R}$,
- concrete reason to distinguish between not UE-bisimilar states.

By application of results from Functional Analysis

- Coalgebra $=$ $\mathbb{R}$-valued Modal Logic
- Coalgebra $=$ Algebra (Riesz space structure)
- Axiomatic approach covers important classes of systems
  - Kripke Structures, Markov Processes, PNTS's, ...
- Expressive logics capable of expressing useful properties (e.g., PCTL) and having good algebraic properties.

Proof Systems?

Abelian Logic = Logic of $(\mathbb{R}, +, -, \sqcup)$

Sequents: $\vdash \phi_1, \ldots, \phi_n$

means $\phi_1 + \cdots + \phi_n \geq 0$ in all interpretations.

Rules:

$$\frac{}{\vdash \phi, -\phi} \qquad \frac{\vdash \Gamma, \phi, \psi}{\vdash \Gamma, \phi + \psi} \qquad \frac{\vdash \Gamma, \phi \qquad \vdash \Gamma, \psi}{\vdash \Gamma, \phi \sqcup \psi}$$

THANKS